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Venkatraman, Aneesh; Ortega, Romeo; Sarras, Ioannis; Schaft, Arjan van der

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# Speed Observation and Position Feedback Stabilization of Partially Linearizable Mechanical Systems

Aneesh Venkatraman, Romeo Ortega, *Fellow, IEEE*, Ioannis Sarras, *Member, IEEE*, and Arjan van der Schaft, *Fellow, IEEE*

**Abstract**—The problems of speed observation and position feedback stabilization of mechanical systems are addressed in this paper. Our interest is centered on systems that can be rendered linear in the velocities via a (partial) change of coordinates. It is shown that the class is fully characterized by the solvability of a set of partial differential equations (PDEs) and strictly contains the class studied in the existing literature on linearization for speed observation or control. A reduced order globally exponentially stable observer, constructed using the immersion and invariance methodology, is proposed. The design requires the solution of another set of PDEs, which are shown to be solvable in several practical examples. It is also proven that the full order observer with dynamic scaling recently proposed by Karagiannis and Astolfi obviates the need to solve the latter PDEs. Finally, it is shown that the observer can be used in conjunction with an asymptotically stabilizing full state-feedback interconnection and damping assignment passivity-based controller preserving asymptotic stability.

**Index Terms**—Output feedback and observers, underactuated mechanical systems.

## I. INTRODUCTION

THE problems of velocity reconstruction and position feedback stabilization (either for regulation or tracking) of mechanical systems are of great practical interest and have been extensively studied in the literature. Since the publication of the first result in the fundamental paper [1] in 1990, many interesting partial solutions have been reported—the reader is referred to the recent books [2]–[4] for an exhaustive list of references.

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A. Venkatraman and A. van der Schaft are with the Institute of Mathematics and Computing Science, University of Groningen, Groningen 9700 AV, The Netherlands (e-mail: aneesh@math.rug.nl; a.j.van.der.schaft@math.rug.nl).

R. Ortega and I. Sarras are with the Laboratoire des Signaux et Systèmes, Supélec, Gif-sur-Yvette 91192, France (e-mail: ortega@lss.supelec.fr; sarras@lss.supelec.fr).

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In this paper these problems are studied for  $n$  degree of freedom mechanical systems modeled in Hamiltonian form as

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{bmatrix} 0 & I_n \\ -I_n & 0 \end{bmatrix} \begin{pmatrix} \frac{\partial H}{\partial q} \\ \frac{\partial H}{\partial p} \end{pmatrix} + \begin{pmatrix} 0 \\ G(q) \end{pmatrix} u \quad (1)$$

where  $q, p \in \mathbb{R}^n$  are the generalized positions and momenta, respectively,  $u \in \mathbb{R}^m$  is the control input,  $m \leq n$  and  $G : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times m}$  is a full rank matrix. The Hamiltonian function  $H : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}$  is the total energy of the system and given as

$$H(q, p) = \frac{1}{2} p^\top M^{-1}(q) p + V(q) \quad (2)$$

where  $M : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  is the mass matrix and  $V : \mathbb{R}^n \rightarrow \mathbb{R}$  is the potential energy function, with  $\mathbb{R}^{n \times n}$  being the set of  $n \times n$  positive definite matrices. It is assumed that the mechanical system does not have any friction effects. In [5] it is shown that it is possible to include friction forces of the form  $R(q)\dot{q}$ , with  $R = R^\top \geq 0$ . In this case, the proposed observer incorporates a term that requires the knowledge of  $R$ . Since in many applications friction forces are negligible and, in any case, they are usually highly uncertain, we have decided to present here the observer for frictionless systems.

The problem is formulated as follows. We assume that *only*  $q$  is measurable and that the input signal  $u(t)$  is such that the system (1) is forward complete, that is, trajectories exist for all  $t \geq 0$ . Our first objective is to design an asymptotically convergent observer for  $p$ . The second objective is to prove that the observer can be used in conjunction with the interconnection and damping assignment passivity-based controller IDA-PBC [6], [7] preserving asymptotic stability by assuming the existence of a full state feedback (IDAPBC) that asymptotically stabilizes a desired equilibrium point  $(q_*, 0)$ .

Attention is centered on mechanical systems that can be rendered linear in the unmeasurable states via a change of coordinates of the form  $(q, P) = (q, \Psi^\top(q)p)$ , with  $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  full rank. The class of systems that satisfy this property, which is fully determined by the inertia matrix  $M$ , will be called in the sequel “Partially Linearizable via Coordinate Changes” (PLvCC). As illustrated in [8]–[12], achieving linearity in  $P$  simplifies the observation as well as the control problem. Unfortunately, the class of mechanical systems considered in the literature is only a small subset of all PLvCC systems—this imposes quite restrictive assumptions on  $M$  and renders their results of limited practical interest. In contrast to this situation, a complete characterization of PLvCC systems, in terms of solvability of a set

of PDEs, is given in this paper. It is furthermore shown that the class contains many examples of practical interest.

For PLvCC systems a globally (exponentially) convergent reduced order immersion and invariance (I&I) observer [2] is proposed in the paper. The design imposes an integrability condition, which is tantamount to the solution of a second set of PDEs. A systematic procedure to solve these PDEs is also given here and its application illustrated with several practical examples. Furthermore, it is shown that the integrability condition can be obviated using the full order I&I observer with dynamic scaling recently proposed in [8]. However, the prize paid for this relaxation is a significant increase in complexity and the, potentially harmful, injection of high gain. A final contribution of our work is the proof that the proposed observer solves the position feedback stabilization problem mentioned above.

The remaining part of the paper is organized as follows. In Section II the characterization of PLvCC systems is given. Section III presents the observer design for this class under the aforementioned integrability assumption. Some subsets of the class of PLvCC systems, and their relation with the systems studied in the linearization literature, are identified in Section IV. Section V presents a constructive procedure to solve the PDEs associated to the integrability assumption. Section VI shows that the proposed observer can be used with IDA-PBC. Some simulation results are given in Section VII. The paper is wrapped-up with some concluding remarks and future work in Section VIII. To enhance readability, some of the (more technical) proofs of the claims are given in Appendices.

*Notation:* For any matrix  $A \in \mathbb{R}^{n \times n}$ ,  $A_i \in \mathbb{R}^n$  denotes the  $i$ -th column,  $A^i$  the  $i$ -th row and  $A_{ij}$  the  $ij$ -th element. That is, with  $e_i, i \in \bar{n} := \{1, \dots, n\}$ , the Euclidean basis vectors,  $A_i := Ae_i$ ,  $A^i := e_i^\top A$  and  $A_{ij} := e_i^\top Ae_j$ .

## II. CHARACTERIZATION OF THE CLASS OF PLvCC SYSTEMS

In this section, we identify the class of mechanical systems for which a change of coordinates of the form  $(q, P) = (q, \Psi^\top(q)p)$ , with  $\Psi$  full rank—uniformly in  $q$ —renders the system linear in  $P$ . As shown below, this property is uniquely defined by the mass matrix  $M$ . The following assumption, which defines a set of PDEs in the unknown  $\Psi$ , is needed.

*Assumption 1:* There exists a full rank matrix  $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  such that, for  $i \in \bar{n}$

$$\mathcal{B}_{(i)}(q) + \mathcal{B}_{(i)}^\top(q) = 0 \quad (3)$$

where the matrices  $\mathcal{B}_{(i)} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  are defined as

$$\mathcal{B}_{(i)}(q) := \sum_{j=1}^n \left\{ [\Psi_i, \Psi_j] \Psi_j^\top (M \Psi \Psi^\top)^{-1} + \frac{1}{2} \Psi_{ji} \Psi \frac{\partial}{\partial q_j} (\Psi^\top M \Psi)^{-1} \Psi^\top \right\} \quad (4)$$

with  $[\Psi_i, \Psi_j]$  being the standard Lie bracket.<sup>1</sup> In this case, the mechanical system (1) is said to be PLvCC—for short, we say that  $M \in \mathcal{S}_{\text{PLvCC}} \subset \mathbb{R}_P^{n \times n}$ .

<sup>1</sup>A standard Lie Bracket of two vector fields  $\Psi_i, \Psi_j$  is defined as  $[\Psi_i, \Psi_j] := (\partial \Psi_j / \partial q) \Psi_i - (\partial \Psi_i / \partial q) \Psi_j$ .

<sup>2</sup>More precisely,  $\mathcal{S}_{\text{PLvCC}}$  is a subset of all mappings  $\mathbb{R}^n \rightarrow \mathbb{R}_P^{n \times n}$ . To avoid cluttering, and with some abuse of notation, in the sequel we will denote  $M \in \mathcal{S}_{\text{PLvCC}}$ .

*Proposition 1:* The dynamics of (1) expressed in the coordinates  $(q, P)$ , where  $P = \Psi^\top(q)p$ , is linear in  $P$  if and only if  $M \in \mathcal{S}_{\text{PLvCC}}$ . In which case, the dynamics becomes

$$\dot{q} = (\Psi^\top M)^{-1} P, \quad \dot{P} = -\Psi^\top \left( \frac{\partial V}{\partial q} - Gu \right). \quad (5)$$

*Proof:* The equation for  $\dot{q}$  follows trivially from the definition of  $P$ . Now,  $\dot{P}$  can be expressed as

$$\begin{aligned} \dot{P} &= \dot{\Psi}^\top p + \Psi^\top \dot{p} \\ &= -\mathbf{D}_{\Psi(q,p)} - \Psi^\top \left( \frac{\partial V}{\partial q} - Gu \right) \end{aligned} \quad (6)$$

where the parameterized mapping  $\mathbf{D}_{\Psi} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$

$$\mathbf{D}_{\Psi(q,p)} := \Psi^\top \frac{\partial}{\partial q} \left\{ \frac{1}{2} p^\top M^{-1} p \right\} - \dot{\Psi}^\top p \quad (7)$$

has been defined. It will now be shown that each element of the vector  $\mathbf{D}_{\Psi}$  is a quadratic form in  $p$ , that is

$$\mathbf{D}_{\Psi} = \sum_{i=1}^n e_i p^\top \mathcal{B}_{(i)} p \quad (8)$$

that becomes zero for all  $p$  if and only if Assumption 1 is satisfied. For, we compute

$$\begin{aligned} \frac{\partial}{\partial q} \left\{ \frac{1}{2} p^\top M^{-1} p \right\} &= \frac{\partial}{\partial q} \left\{ \frac{1}{2} p^\top \Psi (\Psi^\top M \Psi)^{-1} \Psi^\top p \right\} \\ &= \frac{1}{2} \sum_{i=1}^n e_i \left\{ 2 p^\top \frac{\partial \Psi}{\partial q_i} (\Psi^\top M \Psi)^{-1} \Psi^\top p \right. \\ &\quad \left. + p^\top \Psi \frac{\partial}{\partial q_i} \left( (\Psi^\top M \Psi)^{-1} \right) \Psi^\top p \right\} \\ &= \sum_{i=1}^n e_i \left\{ p^\top \frac{\partial \Psi}{\partial q_i} \Psi^{-1} M^{-1} p \right. \\ &\quad \left. + \frac{1}{2} p^\top \Psi \frac{\partial}{\partial q_i} \left( (\Psi^\top M \Psi)^{-1} \right) \Psi^\top p \right\}. \end{aligned} \quad (9)$$

Replacing (9) in (7) we obtain

$$\begin{aligned} \mathbf{D}_{\Psi} &= \dot{\Psi}^\top p + \sum_{i=1}^n \left\{ (\Psi^\top e_i) \left( p^\top \frac{\partial \Psi}{\partial q_i} \right) \Psi^{-1} M^{-1} p + \frac{1}{2} (\Psi^\top e_i) \right. \\ &\quad \left. \times \left[ p^\top \Psi \frac{\partial}{\partial q_i} \left( (\Psi^\top M \Psi)^{-1} \right) \Psi^\top p \right] \right\} \\ &= \sum_{i=1}^n \left\{ e_i (p^\top \mathcal{J}_i p) + \frac{1}{2} (\Psi^\top e_i) \right. \\ &\quad \left. \times \left[ p^\top \Psi \frac{\partial}{\partial q_i} \left( (\Psi^\top M \Psi)^{-1} \right) \Psi^\top p \right] \right\} \\ &= \sum_{i=1}^n e_i p^\top \mathcal{B}_{(i)} p \end{aligned} \quad (10)$$

where we use Lemma 2 from Appendix A to get the first term in the second equation of (10) and the definition of  $\mathcal{B}_{(i)}$  given in (4) for the latter. Hence, the proof follows. ■

*Remark 1:* When Assumption 1 does not hold, the transformed dynamics in the coordinates  $(q, P)$  is given by

$$\begin{pmatrix} \dot{q} \\ \dot{P} \end{pmatrix} = \begin{bmatrix} 0 & \Psi \\ -\Psi^\top & \mathcal{J} \end{bmatrix} \begin{pmatrix} \frac{\partial \tilde{H}}{\partial q} \\ \frac{\partial \tilde{H}}{\partial P} \end{pmatrix} + \begin{pmatrix} 0 \\ \Psi^\top G \end{pmatrix} u \quad (11)$$

with the new energy function being

$$\bar{H}(q, P) := \frac{1}{2}P^\top(\Psi^\top M\Psi)^{-1}P + V(q)$$

and the  $jk$  element of the skew-symmetric matrix  $\mathcal{J} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  being given by

$$\mathcal{J}_{jk}(q, p) = -p^\top[\Psi_j, \Psi_k]. \quad (12)$$

The proof, requiring some cumbersome calculations, is given in Appendix B. See also (62) in Appendix A and [13].

### III. IMMERSION AND INVARIANCE OBSERVERS FOR PLvCC SYSTEMS

#### A. Problem Formulation and Proposed Approach

In this note the observer design framework proposed in [14], which follows the I&I principles first articulated in [15]—see [2] for a tutorial account of this method and its applications—is adopted. In the context of observer design the objective of I&I is to generate an attractive invariant manifold, defined in the extended state-space of the plant and the observer. This manifold is defined by an invertible function in such a way that the unmeasurable part of the state can be reconstructed by inversion of this function.

*Definition 1:* The dynamical system

$$\dot{\eta} = \alpha(q, \eta, u) \quad (13)$$

with  $\eta \in \mathbb{R}^n$ , is called a reduced order I&I observer for the system (1) if there exists a full rank matrix  $\Psi : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  and a vector function  $\beta : \mathbb{R}^n \rightarrow \mathbb{R}^n$ , such that the manifold

$$\mathcal{M} := \{(\eta, q, p) : \beta(q) = \eta + \Psi^\top(q)p\} \subset \mathbb{R}^n \times \mathbb{R}^n \times \mathbb{R}^n \quad (14)$$

is invariant and attractive,<sup>3</sup> with respect to the system (1), (13). The asymptotic estimate of  $p$ , denoted by  $\hat{p}$ , is then given by

$$\hat{p} = \Psi^{-\top}(\beta - \eta).$$

*Remark 2:* The manifold  $\mathcal{M}$  in (14) is a particular case of the one considered in [14], where it is defined as  $\{(\eta, q, p) : \hat{\beta}(q, \eta) = \Psi^\top(q)p\}$ , with  $\hat{\beta} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$ —notice that  $\hat{\beta}$  is a linear function of  $\eta$  in (14). It is clear that, by considering a more general manifold expression it is possible—in principle—to handle a larger class of systems. However, for the purpose of our work, which is to explicitly define (in terms of PDEs) a class of inertia matrices for which the construction works, this is done without loss of generality—see Remark 8 for some additional relationships between both observers.

#### B. A Globally Exponentially Convergent Reduced Order I&I Observer for PLvCC Systems

To present the proposed observer the following assumption is needed.

*Assumption 2:* There exists a mapping  $\beta : \mathbb{R}^n \rightarrow \mathbb{R}^n$  satisfying the matrix inequality

$$\mathcal{Q}\mathcal{A}(q) + \mathcal{A}^\top(q)\mathcal{Q} \geq \epsilon I_n \quad (15)$$

<sup>3</sup>We recall that the set  $\mathcal{M}$  is invariant if  $(\eta(0), q(0), p(0)) \in \mathcal{M} \Rightarrow (\eta(t), q(t), p(t)) \in \mathcal{M}$  for all  $t \geq 0$ . It is said to be globally attractive if, for all  $(\eta(0), q(0), p(0))$ , the distance of the state vector to the manifold, that is,  $\text{dist}\{(\eta, q, p), \mathcal{M}\} = 0$ , asymptotically goes to zero where  $\text{dist}\{x, S\} = \inf\{\text{dist}(x, s) : s \in S\}$  for any set  $S$ .

uniformly in  $q$ , for some  $\epsilon > 0$  and some constant matrix  $\mathcal{Q} \in \mathbb{R}^{n \times n}$ , where

$$\mathcal{A}(q) := \frac{\partial \beta}{\partial q}(q) [\Psi^\top(q)M(q)]^{-1}. \quad (16)$$

*Proposition 2:* Consider the mechanical system (1). Assume  $M \in \mathcal{S}_{\text{PLvCC}}$  with a matrix  $\Psi$  whose inverse is uniformly bounded and that there exists a mapping  $\beta$  satisfying Assumption 2. Then, the dynamical system

$$\begin{aligned} \dot{\eta} &= \frac{\partial \beta}{\partial q}(q)(\Psi^\top M)^{-1}(\beta - \eta) + \Psi^\top \left( \frac{\partial V}{\partial q} - Gu \right) \\ \dot{p} &= \Psi^{-\top}(\beta - \eta) \end{aligned} \quad (17)$$

is a globally exponentially convergent reduced order I&I observer—with the estimation error verifying

$$|\hat{p}(t) - p(t)| \leq \alpha \exp^{-\rho t} |\hat{p}(0) - p(0)|$$

for some  $\alpha, \rho > 0$ , where  $|\cdot|$  is the Euclidean norm.

*Proof:* By following the I&I procedure [2], we prove that the manifold  $\mathcal{M}$ , defined in (14), is attractive and invariant by showing that the off-the-manifold coordinate

$$z = \beta - \eta - \Psi^\top p = \beta - \eta - P \quad (18)$$

verifies: (i)  $(z(0) = 0 \Rightarrow z(t) = 0)$  for all  $t \geq 0$ , and (ii)  $z(t)$  asymptotically (actually, exponentially) converges to zero. Note that  $\text{dist}\{(\eta, q, p), \mathcal{M}\} = 0$  if and only if  $z = 0$ .

To obtain the dynamics of  $z$ , we differentiate (18) to get

$$\begin{aligned} \dot{z} &= \dot{\beta} - \dot{\eta} - \dot{P} \\ &= \frac{\partial \beta}{\partial q} M^{-1} p - \frac{\partial \beta}{\partial q}(q)(\Psi^\top M)^{-1}(\beta - \eta) \\ &= -\mathcal{A}z \end{aligned}$$

where (5) and (17) are used for the second identity while the third one is obtained invoking (16) and (18).

The manifold  $\mathcal{M}$  is clearly positively invariant. To establish global exponential attractivity of  $\mathcal{M}$ , consider the Lyapunov function

$$\mathcal{V}(z) = \frac{1}{2}z^\top \mathcal{Q}z. \quad (19)$$

Condition (15) ensures that

$$\dot{\mathcal{V}} \leq -\frac{\epsilon}{\bar{\lambda}(\mathcal{Q})}\mathcal{V} \quad (20)$$

with  $\bar{\lambda}(\mathcal{Q})$  denoting the maximum eigenvalue of  $\mathcal{Q}$ , which proves, after some basic bounding, the global exponential convergence to zero of  $z$ . Exponential convergence of  $\hat{p} - p$  is concluded invoking uniform boundedness of  $\Psi^{-1}$ . ■

*Remark 3:* Assumption 2 may be rephrased as follows. Assume there exists a mapping  $\mathcal{N} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  such that (15) holds with

$$\mathcal{A}(q) = \mathcal{N}(q) [\Psi^\top(q)M(q)]^{-1} \quad (21)$$

and

$$\frac{\partial \mathcal{N}^j}{\partial q} = \left( \frac{\partial \mathcal{N}^j}{\partial q} \right)^\top, \quad j \in \bar{n}. \quad (22)$$

The latter (integrability) condition ensures, from Poincaré's Lemma, that there exists a  $\beta$  such that

$$\frac{\partial \beta}{\partial q} = \mathcal{N}. \quad (23)$$

That is, the problem reduces to the solution of the PDE (22), subject to the inequality constraint (15),(21). In order to avoid the hard problem of solving constrained PDEs, in Section V a step-by-step procedure to compute  $\mathcal{N}$ , that involves the solution of simpler unconstrained PDEs, is proposed.

### C. Full Order I&I Observer with Dynamic Scaling

In the recent interesting paper [8], a full order I&I observer for a class of nonlinear systems has been proposed that obviates the integrability condition imposed on  $\beta$  in Assumption 2. PLvCC mechanical systems written in the form (5) belong to this class. For the sake of comparison with our work, we present here the observer that results from the application of the techniques in [8] to the present problem.<sup>4</sup>

**Proposition 3:** Consider the mechanical system (1). Assume  $M \in \mathcal{S}_{\text{PLvCC}}$  for some matrix  $\Psi$ . Fix a matrix  $\mathcal{N} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  satisfying

$$\mathcal{N}M^{-1}\Psi^{-\top} + \Psi^{-1}M^{-1}\mathcal{N}^{\top} \geq \epsilon I_n \quad (24)$$

for some  $\epsilon > 0$ . Define the full order I&I observer with dynamic scaling

$$\begin{aligned} \dot{\eta} &= \frac{\partial \tilde{\beta}}{\partial q} (\Psi^{\top} M)^{-1} (\tilde{\beta} - \eta) + \Psi^{\top} \left( \frac{\partial V}{\partial q} - Gu \right) + \frac{\partial \tilde{\beta}}{\partial \dot{q}} \dot{q} \\ \dot{\tilde{q}} &= (\Psi^{\top} M)^{-1} (\tilde{\beta} - \eta) - K(q, \tilde{q}, r) \tilde{q} \\ \dot{r} &= -\epsilon(r-1) + cr \sum_{j=1}^n \tilde{q}_j^2 \|\Delta_j(q, \tilde{q})\|^2, \quad r(0) = 1 \end{aligned}$$

where  $c > 0$ ,  $\|\cdot\|$  is the induced 2-norm

$$\begin{aligned} \tilde{q} &:= \hat{q} - q \\ \tilde{\beta}(q, \hat{q}) &:= \int_0^{q_1} \mathcal{N}_1(\sigma, \hat{q}_2, \dots, \hat{q}_n) d\sigma + \dots \\ &\quad \dots + \int_0^{q_n} \mathcal{N}_n(\hat{q}_1, \hat{q}_2, \dots, \sigma) d\sigma \\ \Delta_j(q, \tilde{q}) &:= [\delta_{1j}(q, \tilde{q}), \dots, \delta_{nj}(q, \tilde{q})], \quad j \in \bar{n} \\ K(q, \tilde{q}, r) &:= cr^2 \left( I_n + \text{diag} \left\{ \|\Delta_j(q, \tilde{q})\|^2 \right\} \right) \end{aligned}$$

where the vector functions  $\delta_{ij} : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  are defined as

$$\begin{aligned} \mathcal{N}_1(q_1, \hat{q}_2, \dots, \hat{q}_n) &= \mathcal{N}_1(q) - \sum_{j=1}^n \tilde{q}_j \delta_{1j}(q, \tilde{q}) \\ &\quad \vdots \\ \mathcal{N}_n(\hat{q}_1, \hat{q}_2, \dots, q_n) &= \mathcal{N}_n(q) - \sum_{j=1}^n \tilde{q}_j \delta_{nj}(q, \tilde{q}). \end{aligned}$$

(i) All signals are bounded.

<sup>4</sup>Strict positivity, as in (15), is imposed below to simplify the presentation. The interested reader is referred to [8] for another, weaker, alternative.

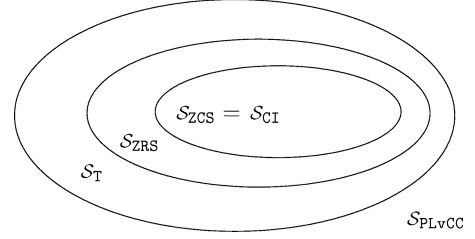


Fig. 1 Sets of inertia matrices with non-decreasing cardinality.

(ii) The system  $(\tilde{z}, \tilde{q})$ , with

$$\tilde{z} = \frac{1}{r}(\tilde{\beta} - \eta - P)$$

has a uniformly globally stable equilibrium at zero.

(iii) The estimation errors  $\tilde{q}(t)$  and  $\tilde{P}(t) := \tilde{\beta}(t) - \eta(t) - P(t)$  converge to zero exponentially fast.

**Remark 4:** Three important features should be considered when comparing the observer given above and the reduced order observer of Proposition 2. First, the complexity of the former is clearly higher than the latter—not just in the dimension of the observer, but also in the number of calculations that are required. Second, although there always exist matrices  $\mathcal{N}$  satisfying (24), it is not always possible to obtain explicit expression for the integrals that define the function  $\tilde{\beta}$ , making the choice of  $\mathcal{N}$  a non-trivial task. These two points are illustrated in the example of Section VII. Third, similar to all designs based on dynamic scaling, the observer given above relies on the injection of *high gain* into the loop—through the function  $r$ —that may be undesirable in some applications. As explained in [8], the leakage factor  $-\epsilon(r-1)$  introduced in the dynamics of  $r$ , is aimed at (partially) alleviating this drawback. In essence, the dynamic scaling design of [8] is a classical trade-off between performance (achievable when the PDE can be solved) and robustness (to dominate via high gain the terms induced by the approximation of the PDE.)

## IV. HOW LARGE IS THE SET $\mathcal{S}_{\text{PLvCC}}$ ?

A natural question that arises at this point is: Under what conditions on  $M$  is Assumption 1 satisfied? Providing a complete answer is tantamount to characterizing all solutions of the PDEs (3), (4), which is clearly a daunting task. It turns out, however, that this set contains some interesting subsets that have a clear physical (and, sometimes, geometric) interpretation—some of which have been studied in the literature, that is briefly reviewed in this section.

### A. Four Subsets of the Set $\mathcal{S}_{\text{PLvCC}}$

To get a better understanding of Assumption 1, four sets of non-decreasing cardinality (displayed in Fig. 1), are shown to be subsets of  $\mathcal{S}_{\text{PLvCC}}$ . Three of them are well-known, but the fourth (and far more interesting) one does not seem to have been reported in the literature. Before presenting these sets we introduce the following important definitions that will be used repeatedly in the sequel.

**Definition 2:** The (full rank) matrix  $T : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  is said to be a factor of  $M^{-1}$  if

$$M^{-1}(q) = T(q)T^{\top}(q). \quad (25)$$

up

*Definition 3:*(i) (*Constant inertia*)

$$\mathcal{S}_{\text{CI}} := \{M \in \mathbb{R}_P^{n \times n} | M_{ij} = \text{constant}, i, j \in \bar{n}\}$$

(ii) (*Zero Christoffel symbols*)

$$\mathcal{S}_{\text{ZCS}} := \{M \in \mathbb{R}_P^{n \times n} | C_{ijk} = 0, i, j, k \in \bar{n}\}$$

where  $C_{ijk} : \mathbb{R}^n \rightarrow \mathbb{R}$  are the Christoffel symbols of the first kind defined as

$$C_{ijk}(q) = \frac{1}{2} \left\{ \frac{\partial M_{ik}}{\partial q_j} + \frac{\partial M_{jk}}{\partial q_i} - \frac{\partial M_{ij}}{\partial q_k} \right\}. \quad (26)$$

(iii) (*Zero Riemann symbols*)

$$\mathcal{S}_{\text{ZRS}} := \{M \in \mathbb{R}_P^{n \times n} | R_{ijkl} = 0, i, j, l, k \in \bar{n}\}$$

with  $R_{ijkl} : \mathbb{R}^n \rightarrow \mathbb{R}$  the Riemann symbols given by

$$R_{ijkl}(q) := \frac{1}{2} \left[ \frac{\partial^2 M_{ik}}{\partial q_j \partial q_l} + \frac{\partial^2 M_{jl}}{\partial q_i \partial q_k} - \frac{\partial^2 M_{il}}{\partial q_j \partial q_k} - \frac{\partial^2 M_{jk}}{\partial q_i \partial q_l} \right] + \sum_{a,b=1}^n (M^{-1})_{ab} [C_{jla} C_{ikb} - C_{ila} C_{jkb}]. \quad (27)$$

(iv) (*Skew-symmetry condition*)

$$\mathcal{S}_{\text{T}} := \{M \in \mathbb{R}_P^{n \times n} | M^{-1} \text{ admits a factor } T \text{ such that} \\ \sum_{j=1}^n [T_i, T_j] T_j^\top = - \left[ \sum_{j=1}^n [T_i, T_j] T_j^\top \right]^\top, i \in \bar{n} \}. \quad (28)$$

*Proposition 4:* The sets of inertia matrices of Definition 3 satisfy

$$\mathcal{S}_{\text{CI}} = \mathcal{S}_{\text{ZCS}} \subset \mathcal{S}_{\text{ZRS}} \subset \mathcal{S}_{\text{T}} \subseteq \mathcal{S}_{\text{PLVCC}}$$

where the inclusion  $\mathcal{S}_{\text{ZCS}} \subset \mathcal{S}_{\text{ZRS}}$  is *strict* for every  $n > 1$ , and the inclusion  $\mathcal{S}_{\text{ZRS}} \subset \mathcal{S}_{\text{T}}$  is *strict* for every  $n > 2$ .

*Proof:*

- ( $\mathcal{S}_{\text{CI}} = \mathcal{S}_{\text{ZCS}}$ ) The fact that  $[M \in \mathcal{S}_{\text{CI}} \Rightarrow M \in \mathcal{S}_{\text{ZCS}}]$  follows trivially from the definition of the Christoffel symbols (26). The proof for  $[M \in \mathcal{S}_{\text{ZCS}} \Rightarrow M \in \mathcal{S}_{\text{CI}}]$  can be worked out by equating all the Christoffel symbols defined in (26) to zero which yields a set of simple PDE's. By performing some straightforward computations, we get to conclude that  $M$  has to be constant.
- ( $\mathcal{S}_{\text{ZCS}} \subset \mathcal{S}_{\text{ZRS}}$ ) The proof that  $[M \in \mathcal{S}_{\text{ZCS}} \Rightarrow M \in \mathcal{S}_{\text{ZRS}}]$  follows from the identity  $\mathcal{S}_{\text{CI}} = \mathcal{S}_{\text{ZCS}}$  and the definition of the Riemann symbols (27). To show that the inclusion is strict, we first recall a well known characterization of the set  $\mathcal{S}_{\text{ZRS}}$ , which may be found in [9], [11], [16]

$$R_{ijkl} = 0, i, j, l, k \in \bar{n} \Leftrightarrow M^{-1} \text{ admits a factor } T \\ \text{such that } [T_i, T_j] = 0, i, j \in \bar{n}. \quad (29)$$

Consider now the physical example of the inverted pendulum on cart depicted in Fig. 3 which has the inertia matrix

$$M = \begin{bmatrix} 1 & b \cos q_1 \\ b \cos q_1 & m_3 \end{bmatrix}, \quad m_3, b > 0. \quad (30)$$

The lower triangular Cholesky factor<sup>5</sup> of  $M^{-1}$  is given by

$$T = \begin{bmatrix} \frac{\sqrt{m_3}}{\sqrt{m_3 - b^2 \cos^2 q_1}} & 0 \\ \frac{-b \cos q_1}{\sqrt{m_3} \sqrt{m_3 - b^2 \cos^2 q_1}} & \frac{1}{\sqrt{m_3}} \end{bmatrix} \quad (31)$$

and it can be easily verified that  $[T_1, T_2] = 0$ . Hence, from (29), the matrix (30) has zero Riemann symbols. We next compute the Christoffel symbols for  $M$  and obtain that  $C_{112} = -b \sin q_1$ , while the rest of the symbols are identically zero. Thus, the inclusion  $\mathcal{S}_{\text{ZCS}} \subset \mathcal{S}_{\text{ZRS}}$  is strict.

- ( $\mathcal{S}_{\text{ZRS}} \subset \mathcal{S}_{\text{T}}$ ) If the columns of  $T$  commute, that is, if  $[T_i, T_j] = 0$ , it is clear that the skew-symmetry condition (28) is satisfied. Hence, by using the equivalence (29), the claim  $[M \in \mathcal{S}_{\text{ZRS}} \Rightarrow M \in \mathcal{S}_{\text{T}}]$  follows in a straight forward manner.

We now proceed to prove that, for  $n > 2$ , the converse implication is not true, which shows that the inclusion is strict. First, we prove that for  $n \leq 2$  the sets are the same. For  $n = 1$  the equivalence is, of course, trivial. For  $n = 2$  this can be easily shown as follows. The skew-symmetry condition (28) yields two equations of the form

$$[T_1, T_2] d^\top = \begin{bmatrix} 0 & \alpha \\ -\alpha & 0 \end{bmatrix}, \quad \alpha \in \mathbb{R} \quad (32)$$

for  $d = T_1, T_2$ , respectively which have a solution if and only if  $[T_1, T_2] = 0$  or  $d = 0$ . The proof follows by noting that since,  $T$  is full rank,  $[T_1, T_2] = 0$  for (32) to hold true. For  $n > 2$  we construct now an inertia matrix  $M \in \mathcal{S}_{\text{T}}$  such that  $M \notin \mathcal{S}_{\text{ZRS}}$ . Towards this end, set  $n = 3$  and consider

$$M^{-1} = \begin{bmatrix} 1 + q_2^2 & 0 & q_2 \sqrt{1 + q_2^2} \\ 0 & (1 + q_2^2)^2 & 0 \\ q_2 \sqrt{1 + q_2^2} & 0 & 1 + q_2^2 \end{bmatrix}. \quad (33)$$

Computing the Riemann symbols and recalling that, because of the symmetries of the tensor, only  $R_{1212}, R_{1213}, R_{1223}, R_{1313}, R_{1323}, R_{2323}$  need to be calculated, one can verify that  $R_{1212}, R_{1323}, R_{2323} \neq 0$  for all  $q$  and  $R_{1223} \neq 0$  for  $q_2 \neq 0$ , and hence conclude that  $M \notin \mathcal{S}_{\text{ZRS}}$ .

We now compute a factor of  $M^{-1}$  verifying (25), given by

$$T = \begin{bmatrix} \sin(q_1) q_2 & \cos(q_1) q_2 & 1 \\ (1 + q_2^2) \cos(q_1) & -(1 + q_2^2) \sin(q_1) & 0 \\ \sqrt{1 + q_2^2} \sin(q_1) & \sqrt{1 + q_2^2} \cos(q_1) & 0 \end{bmatrix}. \quad (34)$$

<sup>5</sup>It can be shown that, since  $M$  is positive definite, this factorization always exists, is uniquely defined and has positive diagonal entries see, e.g., Corollary 7.2.9 of [17] and [18].

Computing the Lie brackets with the vectors  $T_i$  one obtains

$$[T_1, T_2] = T_3, \quad [T_2, T_3] = T_1, \quad [T_3, T_1] = T_2. \quad (35)$$

Hence, each of the matrices  $[T_1, T_2]T_2^\top + [T_1, T_3]T_3^\top$ ,  $[T_2, T_1]T_1^\top + [T_2, T_3]T_3^\top$  and  $[T_3, T_1]T_1^\top + [T_3, T_2]T_2^\top$  are skew symmetric as desired. This completes the proof.

- ( $\mathcal{S}_T \subseteq \mathcal{S}_{\text{PLVCC}}$ ) It will be shown that if  $M \in \mathcal{S}_T$ , Assumption 1 is satisfied with  $\Psi = T$ . Indeed, replacing  $\Psi = T$  in (4), the second right term vanishes and we get

$$\mathcal{B}_{(i)} = \sum_{j=1}^n [T_i, T_j]T_j^\top. \quad (36)$$

Now, the skew-symmetry condition (28) and (36) ensure that the condition (3) is satisfied. Hence the proof follows. ■

*Remark 5:* The case  $M \in \mathcal{S}_{\text{ZRS}}$  has been extensively studied in analytical mechanics and has a deep geometric significance stemming from Theorem 2.36 in [19]. The property has been exploited, in the context of linearization, in the control literature in [9], [11]. If  $M \in \mathcal{S}_{\text{ZRS}}$  the system is said to be *Euclidean* [9], where the qualifier stems from the fact that the system is diffeomorphic to a “linear double integrator”—see Proposition 6 below.

*Remark 6:* An explanation regarding the construction of the example used in the proof [ $M \in \mathcal{S}_T \not\Rightarrow M \in \mathcal{S}_{\text{ZRS}}$ ] is in order and is given in Appendix C.

### B. Physical Interpretation of the Sets $\mathcal{S}_{\text{ZCS}}$ , $\mathcal{S}_{\text{ZRS}}$ , $\mathcal{S}_T$ and Implications for the Observer Design

For which classes of physical systems the mass matrix belongs to the sets of Proposition 4? How to select the matrix  $\mathcal{N}$  of Assumption 2 to complete the observer design in those cases? Answers to these questions are provided in this subsection. Clearly, to choose the matrix  $\mathcal{N}$  it is necessary to know the matrix  $\Psi$  that verifies Assumption 1. As shown in the proof of Proposition 1, Assumption 1 holds if and only if the mapping  $\mathbf{D}_\Psi$ , defined in (7), identically vanishes. This test will, therefore, be used to answer the questions. An additional verification to analyze  $\mathbf{D}_\Psi$  is that it allows to establish some connections of our work with the existing literature.

To streamline the presentation in the sequel, it is convenient to recall the Lagrangian model of the mechanical system (1) being given as

$$M(q)\ddot{q} + C(q, \dot{q})\dot{q} + \frac{\partial V}{\partial q} = G(q)u \quad (37)$$

where  $C(q, \dot{q})\dot{q}$  is the vector of Coriolis and centrifugal forces, with the  $ik$ -th element of  $C : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  defined by

$$C_{ik}(q, \dot{q}) = \sum_{j=1}^n C_{ijk}(q)\dot{q}_j.$$

For future reference, we recall the well-known property

$$\dot{M} = C + C^\top. \quad (38)$$

We next relate the key mapping  $\mathbf{D}_\Psi$ , defined in (7), with the matrices  $M$  and  $C$ . For this, we define the vector function  $\tilde{\mathbf{D}}_\Psi : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^n$  as  $\tilde{\mathbf{D}}_\Psi(q, \dot{q}) := \mathbf{D}_\Psi(q, M(q)\dot{q})$ . Hence, from (7) we get

$$\begin{aligned} \tilde{\mathbf{D}}_\Psi &= \Psi^\top \frac{\partial}{\partial q} \left\{ \frac{1}{2} \dot{q}^\top M \dot{q} \right\} - \dot{\Psi}^\top M \dot{q} \\ &= \left[ \Psi^\top C - \frac{d}{dt}(\Psi^\top M) \right] \dot{q} \end{aligned} \quad (39)$$

where, to obtain the second identity, we have used the well-known fact (see, e.g., [20]) that

$$\frac{\partial}{\partial q} \left\{ \frac{1}{2} \dot{q}^\top M \dot{q} \right\} = (\dot{M} - C)\dot{q}. \quad (40)$$

#### 1) The Set $\mathcal{S}_{\text{ZCS}}$ :

*Proposition 5:* The following statements are equivalent.

- (i)  $M \in \mathcal{S}_{\text{ZCS}}$ .
  - (ii) Assumption 1 holds for any constant  $\Psi$ .
  - (iii) The Coriolis and centrifugal forces  $C(q, \dot{q})\dot{q}$  equal zero.
- Moreover, if  $M \in \mathcal{S}_{\text{ZCS}}$ , and we take  $\Psi = M^{-1}$ , the transformed dynamics (5) become

$$\dot{q} = P, \quad \dot{P} = -M^{-1} \left( \frac{\partial V}{\partial q} - Gu \right). \quad (41)$$

*Proof:* The equivalence between (i) and (iii) follows recalling that, for all vectors  $x, y \in \mathbb{R}^n$

$$C(q, x)y = \begin{bmatrix} x^\top C^{(1)}(q)y \\ x^\top C^{(2)}(q)y \\ \vdots \\ x^\top C^{(n)}(q)y \end{bmatrix}$$

where the  $ij$  elements of the symmetric matrix  $C^{(k)} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  are precisely the Christoffel symbols  $C_{ijk}$ . Now, from (7), it is clear that (ii) is true if and only if

$$\frac{\partial}{\partial q} \left\{ \frac{1}{2} p^\top M^{-1} p \right\} = 0$$

which is equivalent to  $M \in \mathcal{S}_{\text{CI}}$ . The proof is completed by recalling from Proposition 4 that  $\mathcal{S}_{\text{ZCS}} = \mathcal{S}_{\text{CI}}$ .

The proof of (41) follows by noting that,  $\Psi = M^{-1}$  gives  $P = \dot{q}$  and  $\dot{P}$  is obtained by replacing  $\mathbf{D}_{M^{-1}} = 0$  in (6). ■

From the definition of  $\mathcal{A}$  in (21) we see that when  $\Psi = M^{-1}$ , Assumption 2 is satisfied with any constant matrix  $\mathcal{N}$  such that  $-\mathcal{N}$  is a Hurwitz matrix. Further, the construction of  $\beta$  from (23) is trivial and the observer error dynamics is linear, namely  $\dot{z} = -\mathcal{N}z$ . For instance, selecting  $\mathcal{N} = I_n$  the observer takes the simple form

$$\dot{\eta} = q - \eta + M^{-1} \left( \frac{\partial V}{\partial q} - Gu \right), \quad \hat{p} = M(q - \eta).$$

The reason why this basic construction works can be easily explained recalling (41) of Proposition 5.

*Remark 7:* In [10] an observer was designed for Lagrangian systems to estimate  $\dot{q}$ , under the following sufficient condition

for linearizability. Define a change of coordinates of the form  $(q, v) = (q, \mathcal{E}(q)\dot{q})$ , with  $\mathcal{E} : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  full rank. Then

$$\begin{aligned} \dot{v} &= \dot{\mathcal{E}}\dot{q} + \mathcal{E}\ddot{q} \\ &= (\dot{\mathcal{E}} - \mathcal{E}M^{-1}C)\mathcal{E}^{-1}v - \mathcal{E}M^{-1}\left(\frac{\partial V}{\partial q} - Gu\right) \end{aligned} \quad (42)$$

where (37) was used to get the last equation. It is clear that the dynamics becomes linear in  $v$  if

$$\dot{\mathcal{E}} = \mathcal{E}M^{-1}C. \quad (43)$$

(Of course, this conclusion also follows from (39) setting  $\Psi = M^{-1}\mathcal{E}^\top$ .) Condition (43) is imposed in [10], which besides being obviously stronger than Assumption 1, does not seem to admit any geometric or system theoretic interpretation.

### 2) The Set $\mathcal{S}_{\text{ZRS}}$ :

*Proposition 6:* The following statements are equivalent:

- (i)  $M \in \mathcal{S}_{\text{ZRS}}$ .
- (ii) There exists a matrix  $T$  which is a factor of  $M^{-1}$  and a mapping  $Q : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that

$$\frac{\partial Q}{\partial q}(q) = T^{-1}(q). \quad (44)$$

Moreover, if  $M \in \mathcal{S}_{\text{ZRS}}$ , Assumption 1 holds with  $\Psi = T$ , that is,  $\mathbf{D}_T = 0$ , and the dynamics expressed in the coordinates  $(Q, P)$  takes the form

$$\dot{Q} = P, \quad \dot{P} = -\frac{\partial \tilde{V}}{\partial Q} + T^\top Gu \quad (45)$$

where  $\tilde{V}(Q) := V(Q^I(Q))$ , with  $Q^I : \mathbb{R}^n \rightarrow \mathbb{R}^n$  a right inverse of  $Q(q)$ , that is,  $Q(Q^I(x)) = x$  for all  $x \in \mathbb{R}^n$ .

*Proof:* See Appendix D. ■

Although the construction of an observer for (45) is trivial—as in the case  $M \in \mathcal{S}_{\text{ZCS}}$ —we underscore that to get the representation (45) it is necessary to solve the PDE (44). This requirement severely restricts the practical applicability of the approach. Indeed, in contrast with the PDEs that are encountered in the paper, the PDE (44) has no *free parameters* and its explicit solution may be even impossible. This is, for instance, the case of the classical cart-pole system. In Section IV-A this system was shown to be Euclidean but, as indicated in [9], (44) leads to an elliptic integral of the second kind that does not admit a closed form.

On the other hand, regarding the reduced order I&I observer, from (16) we see that when  $\Psi = T$  one gets  $\mathcal{A} = \mathcal{N}T$ . In Section V we propose to take  $T$  to be the lower triangular Cholesky factor of  $M^{-1}$ , and present a procedure to design  $\mathcal{N}$  in order to satisfy Assumption 2.

### 3) The Set $\mathcal{S}_T$ :

*Proposition 7:* For any matrix  $T$ , factor of  $M^{-1}$ , the following statements are equivalent:

- (i)  $M \in \mathcal{S}_T$ .
  - (ii) Assumption 1 holds with  $\Psi = T$ , that is,  $\mathbf{D}_T = 0$ .
- Further, if  $M \in \mathcal{S}_T$ , the transformed dynamics takes the form

$$\dot{q} = TP, \quad \dot{P} = -T^\top \left( \frac{\partial V}{\partial q} - Gu \right).$$

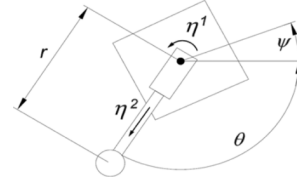


Fig. 2. Robotic leg, where we denote  $q := (r, \theta, \psi)$ .

*Proof:* The evaluation of the matrices  $\mathcal{B}_{(i)}$ , defined in (4), for  $\Psi = T$  is given in (36). Now, from (10) we conclude that  $\mathbf{D}_T = 0$  if and only if these matrices are skew symmetric, which is precisely the condition for  $M \in \mathcal{S}_T$ .

The last claim is established by noting that,  $\Psi = T$  gives  $\dot{q} = TP$  and  $\dot{P}$  is obtained by replacing  $\mathbf{D}_T = 0$  in (6). ■

Similar to the systems belonging to  $\mathcal{S}_{\text{ZRS}}$ , we get in this case  $\mathcal{A} = \mathcal{N}T$  and the procedure for construction of  $\mathcal{N}$  in Section V is applicable.

*Remark 8:* Some connections between our observer and the one proposed in [14] may be established at this point. Towards this end, we refer to the function  $\tilde{\mathbf{D}}_\Psi$  defined in (39) and evaluate it for  $\Psi = T$  to obtain

$$\tilde{\mathbf{D}}_T = \left[ T^\top C - \frac{d}{dt}T^{-1} \right] \dot{q} =: \tilde{C}(q, \dot{q})\dot{q}.$$

It can be shown that the matrix  $\tilde{C}(q, \dot{q})T(q)$  is linear in  $\dot{q}$  and furthermore, invoking (38), we can also prove that it is skew-symmetric. These properties are used in [14] to, adding to the observer a “certainty-equivalent” term  $\tilde{C}(q, T\tilde{\beta})T\tilde{\beta}$ , generate an error dynamics of the form  $\dot{z} = [\Gamma(q, \eta) - \tilde{C}(q, \dot{q})T(q)]z$ , where  $\Gamma : \mathbb{R}^n \times \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  is a matrix that can be shaped by selecting the function  $\tilde{\beta}$ . A constructive solution is given for some particular cases of systems with  $n = 2$ , namely: diagonal inertia matrix (with possible unbounded elements) and inertia matrix with bounded elements. Some recent calculations show that this technique can be extended beyond these cases, but the need to explicitly solve the integrals that define  $\tilde{\beta}$  make this an “existence result”, more than an actual constructive procedure. Of course, it may be argued that the route taken in the present paper (that aims at eliminating the term  $\mathbf{D}_\Psi$ ), although leading to the explicit identification of some PDEs to be solved, is also not constructive—given our inability to guarantee their solution in general.

### C. Robotic Leg: $M \in \mathcal{S}_{\text{PLVCC}}$ But $M \notin \mathcal{S}_{\text{ZRS}}$

Some of the developments presented above are illustrated in this subsection on the robotic leg example [21] depicted in Fig. 2. We have  $n = 3$

$$M = \text{diag} \{m_1, m_1 q_1^2, m_2\} \quad (46)$$

with  $m_1, m_2 > 0$ , and the position restricted to the set  $q \in \{q_1 \geq \epsilon > 0\}$ . Firstly, the only non-zero Christoffel symbols are  $C_{122} = -C_{221} = m_1 q_1$  which implies that  $M \notin \mathcal{S}_{\text{ZCS}}$ . Furthermore, the Riemann symbol  $R_{1212} = m_1 \neq 0$  implies that  $M \notin \mathcal{S}_{\text{ZRS}}$ . We will now prove that  $M \in \mathcal{S}_{\text{PLVCC}}$  provided

$$q \in \mathcal{C} := \{q_1 \geq \epsilon > 0, q_2 \neq i\pi, i \in \mathbb{Z}_+\}.$$



Indeed, some lengthy but straightforward calculations, prove that the matrix

$$\Psi(q_1, q_2) := \begin{bmatrix} \sin(q_2) & \sin(q_2) & 0 \\ \frac{1}{q_1} \cos(q_2) + \kappa & \frac{1}{q_1} \cos(q_2) & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \kappa \neq 0 \quad (47)$$

which is well-defined and full-rank for all  $q \in \mathcal{C}$ , ensures  $\mathbf{D}_\Psi = 0$  for the inertia matrix (46). It should be pointed out that (47) was obtained by solving the PDEs (3), (4) for the inertia matrix (46). For the elements of  $\Psi_1$  they are of the form

$$\frac{\partial \Psi_{11}}{\partial q_1} = 0, \quad \frac{\partial \Psi_{11}}{\partial q_2} + q_1^2 \frac{\partial \Psi_{21}}{\partial q_1} = 0, \quad q_1 \frac{\partial \Psi_{21}}{\partial q_2} = -\Psi_{11}$$

with a similar form for the elements of  $\Psi_2$ .

*Remark 9:* Although not yet proven, some preliminary calculations lead us to conjecture that  $M \notin \mathcal{S}_T$ . Notice that the “natural” choice for the factor  $T$ , namely

$$T = \text{diag} \left\{ \frac{1}{\sqrt{m_1}}, \frac{1}{q_1 \sqrt{m_1}}, \frac{1}{\sqrt{m_2}} \right\}$$

does not satisfy the condition  $[T_1, T_2]T_2^\top = -([T_1, T_2]T_2^\top)^\top$ .

*Remark 10:* It is interesting to note that, in spite of the similarities with the robotic leg, the classical ball-and-beam system is not *PLvCC*. The mass matrix of the ball-and-beam is  $M = \text{diag}\{1, \ell^2 + q_1^2\}$ , where  $\ell > 0$  is the length of the beam, and  $q \in \{|q_1| \leq \ell\}$ . The PDE's for  $\Psi_1$  are

$$\frac{\partial \Psi_{11}}{\partial q_1} = 0, \quad \frac{\partial \Psi_{11}}{\partial q_2} + (\ell^2 + q_1^2) \frac{\partial \Psi_{21}}{\partial q_1} = 0, \quad \frac{\partial \Psi_{21}}{\partial q_2} = \frac{-q_1}{\ell^2 + q_1^2} \Psi_{11}.$$

The first and third PDE's together imply

$$\Psi_{21}(q_1, q_2) = \frac{q_1}{\ell^2 + q_1^2} \tilde{\Psi}_{21}(q_2) + \kappa \quad (48)$$

where  $\Psi_{11}(q_2) = -(\partial \tilde{\Psi}_{21} / \partial q_2)$ . Next, using (48) together with the second PDE yields the ODE

$$\frac{d^2}{dq_2^2} \tilde{\Psi}_{21}(q_2) = \frac{\ell^2 - q_1^2}{\ell^2 + q_1^2} \tilde{\Psi}_{21}(q_2)$$

that, clearly, does not admit a solution.

## V. A CONSTRUCTIVE PROCEDURE FOR $\mathcal{N}$

In this Section we present a simple algorithm to construct a matrix  $\mathcal{N}$  that satisfies Assumption 2. The starting point of the procedure is to compute the lower triangular Cholesky factorization,  $T$  of  $M^{-1}$  and select  $\Psi = T$ . The idea is then to construct a matrix  $\mathcal{N}$  such that, on one hand,  $\mathcal{N}T$  is *diagonal* with positive diagonal entries and, on the other hand,  $\mathcal{N}$  is “trivially” integrated—in the sense of Remark 3. The first condition will ensure (15) of Assumption 2, while the second one guarantees (22). As expected, the construction involves the solution of some PDEs that we show can be easily solved for several examples of practical interest.

We now present the algorithm for computing  $\mathcal{N}$  when the mass matrix depends on  $k$  coordinates where  $k \in \bar{n}$ .

### A. Procedure for Computing $\mathcal{N}$ when $M$ Depends on $k$ Coordinates

Without loss of generality we assume that the mass matrix depends on the first  $k$  coordinates  $q_1, q_2, \dots, q_k$ . Next, we propose the following form for  $\mathcal{N}$  given as

$$\mathcal{N} = \Lambda + \frac{\partial}{\partial q} \{ \phi(q_1, q_2, \dots, q_k) + \psi(q_1, q_2, \dots, q_k)q \} \quad (49)$$

where  $\Lambda > 0$  is an  $n \times n$  constant diagonal matrix,  $\phi : \mathbb{R}^k \rightarrow \mathbb{R}^n$  verifies  $e_1^\top \phi = 0$  and  $\psi : \mathbb{R}^k \rightarrow \mathbb{R}^{n \times n}$  verifies  $e_i^\top \psi e_j = 0$  for all  $j \geq i$  and  $\psi e_1 = \psi e_2 = \psi e_3 = \dots = \psi e_k = 0$ .

Given the proposed form for  $\mathcal{N}$ , (ii) of Assumption 2 is trivially satisfied and  $\beta$  can be immediately computed as

$$\beta = \Lambda q + \phi(q_1, q_2, \dots, q_k) + \psi(q_1, q_2, \dots, q_k)q. \quad (50)$$

We can check from (49) that for  $k = 1, 2$ , the matrix  $\mathcal{N}$  is lower triangular and hence is  $N = \mathcal{N}T$  but, for  $k > 2$ , the  $k \times k$  upper left block of the matrix  $\mathcal{N}$  (and subsequently  $N$ ) is clearly not lower triangular which makes the algorithm more complicated. The algorithm proceeds along the following steps:

1. Compute  $\tilde{N} = (1/2)\{N + N^\top\}$ .
2. For every  $i \geq k + 2$ , solve  $\tilde{N}_{i,i-1} = 0$  to obtain  $\psi_{i,i-1}$ .
3. For every  $i \geq k + 3$ , solve  $\tilde{N}_{i,i-2} = 0$  by using the function  $\psi_{i,i-1}$  obtained in step 1 to get  $\psi_{i,i-2}$ .
4. Proceed in this manner until  $i = n$  to complete the computation of  $\psi$ .
5. Solve the inequalities  $\tilde{N}_{ii} > 0$  for all  $2 \leq i \leq k$  and the partial differential equations  $\tilde{N}_{ij} = 0$  for all  $1 \leq j < i \leq k$  to determine the functions  $\phi_l$ , for all  $l = 2, \dots, k$ .
6. Solve the partial differential equations  $\tilde{N}_{ij} = 0$ ,  $k + 1 \leq i \leq n$ ,  $1 \leq j \leq k$  and compute the matrix  $\phi$ .

The elements of the matrix  $\Lambda$  can be chosen freely and it suffices to just ensure that they are positive constants. Finally, after having computed  $\mathcal{N}$ , we obtain  $\beta$  from (50).

*Remark 11:* Step 5 is the difficult one as it involves solving  $(k - 1)$  inequalities and  $(k)(k - 1)/2$  partial differential equations with the number of unknowns being  $k(k - 1)$ . We can see that for  $k = 1$ , step 5 can be skipped. For  $k = 2$ , the number of equations (inequalities and equalities together) is same as the number of unknowns and we thus get an exact solution, but for  $k > 2$  we have more equations than unknowns. Hence, it could be possible that we can get more than one solution for the functions  $\phi_i$  for  $i = 2, \dots, k$ . The steps 2, 3, in the algorithm, which involve solving a set of algebraic equations and step 6 that involves a simple set of PDE's are relatively straightforward.

*Remark 12:* If  $k = n$ , then the matrix  $\psi = 0$  from our construction. In that case, we would have to follow only step 5 of the algorithm. Hence (as expected), the larger the value of  $k$ , more PDEs need to be solved and the complication of the algorithm increases.

We now illustrate this procedure for a 2-dof and 3-dof system where  $k = 1$  and a 4-dof system where  $k = 2$ .

1) *Inverted Pendulum on a Cart [6]:* The inertia matrix of the well-known inverted pendulum on a cart system depicted in Fig. 3 is given in (30) and the lower triangular Cholesky factor

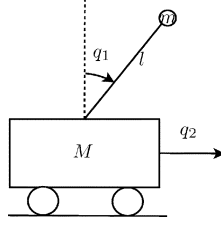


Fig. 3. Cart-pendulum system.

in (31). As shown before  $[T_1, T_2] = 0$  thus satisfying Assumption 1. We now proceed to construct  $\mathcal{N}$  by following the above algorithm and accordingly set it as

$$\mathcal{N} = \begin{bmatrix} \Lambda_{11} & 0 \\ \frac{\partial \phi_2}{\partial q_1} & \Lambda_{22} \end{bmatrix}$$

where  $\Lambda_{ii} > 0$ . We next solve the ordinary differential equation,  $N_{21} = 0$ , which is of the form  $(d\phi_2/dq_1) = (\Lambda_{22}b/m_3) \cos(q_1)$  to get  $\phi_2 = (\Lambda_{22}b/m_3) \sin(q_1)$ . Thus, we obtain

$$\beta = \begin{bmatrix} \Lambda_{11}q_1 \\ \Lambda_{22} \left( q_2 + \frac{b}{m_3} \sin(q_1) \right) \end{bmatrix}. \quad (51)$$

We later show some simulation results for this example in Section VII.

2) *3-Link Underactuated Planar Manipulator* [22]: This is a 3-dof underactuated mechanical system depicted in Fig. 4 with

$$M^{-1} = \frac{1}{F^2} \begin{bmatrix} 1 & \frac{m_3 L}{m_x} \sin q_1 & -\frac{m_3 L}{m_y} \cos q_1 \\ * & \frac{m_y I + m_3^2 L^2}{m_x m_y} \cos^2 q_1 & -\frac{m_3 L^2}{m_x m_y} \sin q_1 \cos q_1 \\ * & * & \frac{m_x I - m_3^2 L^2}{m_x m_y} \sin^2 q_1 \end{bmatrix}$$

where

$$F(q) := \sqrt{1 - (m_3^2 L^2 / m_y) \cos^2 q_1 - (m_3^2 L^2 / m_x) \sin^2 q_1}.$$

We compute the lower triangular Cholesky factorization as

$$T = \begin{bmatrix} \frac{1}{F} & 0 & 0 \\ \frac{m_3 L}{m_x F} \sin q_1 & \frac{1}{\sqrt{m_x}} & 0 \\ -\frac{m_3 L}{m_y F} \cos q_1 & 0 & \frac{1}{\sqrt{m_y}} \end{bmatrix}.$$

We can easily check that the columns of  $T$  commute and thus the system is Euclidean. Following the procedure described above, we set  $\mathcal{N}$  as

$$\mathcal{N} = \begin{bmatrix} \Lambda_{11} & 0 & 0 \\ \frac{\partial \phi_2}{\partial q_1} & \Lambda_{22} & 0 \\ \frac{\partial \psi_{32}}{\partial q_1} q_2 + \frac{\partial \phi_3}{\partial q_1} & \psi_{32} & \Lambda_{33} \end{bmatrix}$$

where  $\Lambda_{ii} > 0$ . We first solve  $N_{32} = 0$  to obtain  $\psi_{32} = 0$ . We next solve  $N_{31} = 0$  and get  $\phi_3 = (\Lambda_{33} m_3 L / m_y) \sin q_1$ . We finally solve  $N_{21} = 0$  to get  $\phi_2 = (\Lambda_{22} m_3 L / m_x) \cos q_1$ . We finally obtain

$$\beta = \begin{bmatrix} \Lambda_{11}q_1 \\ \Lambda_{22} \left( q_2 + \frac{m_3 L}{m_x} \cos q_1 \right) \\ \Lambda_{33} \left( q_3 + \frac{m_3 L}{m_y} \sin q_1 \right) \end{bmatrix}.$$

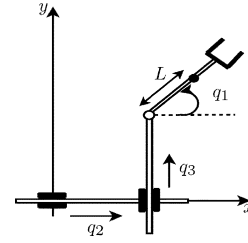


Fig. 4. 3-Link underactuated planar manipulator.

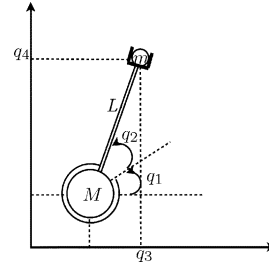


Fig. 5. Planar redundant manipulator with one elastic degree of freedom.

3) *Planar Redundant Manipulator With One Elastic Degree of Freedom* [23]: This is an interesting example of a 4-dof underactuated mechanical system whose mass matrix depends on two coordinates, with  $M^{-1}$  being given as (see Fig. 5)

$$\begin{bmatrix} \frac{1}{I} & -\frac{1}{I} & 0 & 0 \\ * & \frac{M+m}{MmL^2} + \frac{1}{I} & -\frac{1}{mL} \sin(q_1 + q_2) & \frac{1}{mL} \cos(q_1 + q_2) \\ * & * & \frac{1}{M+m} & 0 \\ * & * & * & \frac{1}{M+m} \end{bmatrix}.$$

We now compute the lower triangular cholesky factorization,  $T$  of  $M^{-1}(q)$  as

$$\begin{bmatrix} \frac{1}{\sqrt{I}} & 0 & 0 & 0 \\ -\frac{1}{\sqrt{I}} & \frac{\sqrt{M+m}}{\sqrt{MmL}} & 0 & 0 \\ 0 & -\sqrt{\frac{M}{m}} \frac{1}{\sqrt{M+m}} \sin(q_1 + q_2) & \frac{1}{\sqrt{M+m}} & 0 \\ 0 & \sqrt{\frac{M}{m}} \frac{1}{\sqrt{M+m}} \cos(q_1 + q_2) & 0 & \frac{1}{\sqrt{M+m}} \end{bmatrix}.$$

We can again easily check that the columns of  $T$  commute among each other thus satisfying Assumption 1. We let the matrix  $\mathcal{N}$  be given as

$$\mathcal{N} = \begin{bmatrix} \Lambda_{11} & 0 & 0 & 0 \\ \frac{\partial \phi_2}{\partial q_1} & \frac{\partial \phi_2}{\partial q_2} + \Lambda_{22} & 0 & 0 \\ \frac{\partial \phi_3}{\partial q_1} & \frac{\partial \phi_3}{\partial q_2} & \Lambda_{33} & 0 \\ \frac{\partial \psi_{43}}{\partial q_1} q_3 + \frac{\partial \phi_4}{\partial q_1} & \frac{\partial \psi_{43}}{\partial q_2} q_3 + \frac{\partial \phi_4}{\partial q_2} & \psi_{43} & \Lambda_{44} \end{bmatrix} \quad (52)$$

where  $\Lambda_{ii} > 0$ . From  $N_{21} = 0$ , we get  $(\partial \phi_2 / \partial q_1) = (\partial \phi_2 / \partial q_2)$  and from  $N_{22} > 0$ , we get  $(\partial \phi_2 / \partial q_1) > 0$ . Thus, we let  $\phi_2 = k(q_1 + q_2)$  where  $k > 0$ . We now solve  $N_{43} = 0$  to obtain  $\psi_{43} = 0$ . We then solve  $N_{42} = 0$  to get

$$\phi_4 = \frac{ML\Lambda_{44}}{M+m} \sin(q_1 + q_2) + g(q_1).$$

Finally, from  $N_{41} = 0$ , we get  $(\partial\phi_4/\partial q_1) = (\partial\phi_4/\partial q_2)$  and hence we can set  $g = 0$ . We next solve  $N_{32} = 0$  to obtain

$$\phi_3 = -\frac{ML\Lambda_{33}}{M+m} \cos(q_1 + q_2) + f(q_1).$$

Next, from  $N_{31} = 0$ , we get  $(\partial\phi_3/\partial q_1) = (\partial\phi_3/\partial q_2)$  and hence we can set  $f = 0$ . We finally get

$$\beta = \begin{bmatrix} \Lambda_{11}q_1 \\ \Lambda_{22}q_2 + k(q_1 + q_2) \\ \Lambda_{33} \left( q_3 - \frac{ML}{M+m} \cos(q_1 + q_2) \right) \\ \Lambda_{44} \left( q_4 - \frac{ML}{M+m} \sin(q_1 + q_2) \right) \end{bmatrix}.$$

### B. Computation of $\mathcal{N}$ for a General Non-Cholesky Factorization of the Inertia Matrix

The constructive procedure to compute  $\mathcal{N}$  given above proceeds from the Cholesky factorization of the matrix  $M^{-1}$ . It may happen that this particular factorization does not satisfy the skew-symmetry condition of Proposition 4 but another factorization does—this is so for the mass matrix (33). Moreover, the inertia matrix may not admit a suitable factorization that satisfies the skew-symmetry condition, but we may be able to find a matrix  $\Psi$  that verifies the most general Assumption 1.

To compute  $\mathcal{N}$  we can, of course, combine the two conditions of Assumption 2 to obtain, directly in terms of  $\beta$ , the differential inequality

$$\frac{\partial\beta}{\partial q} M^{-1} \Psi^{-\top} + \left( \frac{\partial\beta}{\partial q} M^{-1} \Psi^{-\top} \right)^\top \geq \epsilon I_n$$

but it seems difficult to even establish conditions for existence of solutions to this inequality. Alternatively, we can fix “candidate” matrices  $\mathcal{N}$  that already satisfy the integrability condition (22) and concentrate on the inequality (15). Obviously, the first natural candidates are constant matrices. Another useful option is to fix the  $ij$  element of  $\mathcal{N}$  to be of the form

$$\mathcal{N}_{ij}(q) = a_{i1}^1(q_1) a_{i2}^2(q_2) \dots \frac{da_{ij} v(q_j)}{dq_j} \dots a_{in}^n(q_n)$$

for some free functions  $a_{ij}^j : \mathbb{R} \rightarrow \mathbb{R}$ —it is easy to see that (22) will hold for the resulting  $\mathcal{N}$ .

We show now how this construction works for the mass matrix (33) with the (non-Cholesky) factorization (34). We recall that, as shown in Proposition 4, the columns of this matrix do not commute, however, it verifies the skew-symmetry condition. For the sake of illustration, we select the desired operating point to be  $q_\star = 0$ .

*Proposition 8:* Consider the matrix  $T$  in (34) and the matrix

$$\mathcal{N} = \begin{bmatrix} 0 & \frac{\lambda}{\sqrt{1+q_2^2}} & 0 \\ 0 & 0 & \lambda \\ \cos(q_1) & 0 & 0 \end{bmatrix}$$

with  $\lambda > 0$ . Then  $\mathcal{N}T + T^\top \mathcal{N}^\top > 0$ , for all  $q$  in the set

$$\left\{ q \in \mathbb{R}^3 \mid \frac{-\pi}{2} + \kappa \leq q_1 \leq \frac{\pi}{2} - \kappa, 4\lambda > |q_2| \right\}$$

where  $\kappa > 0$  is an arbitrarily small constant.

*Proof:* We compute  $\mathcal{N}T + T^\top \mathcal{N}^\top$  as

$$\begin{bmatrix} 2\lambda\sqrt{1+q_2^2}\cos(q_1) & 0 & \frac{1}{2}\sin(2q_1)q_2 \\ * & 2\lambda\sqrt{1+q_2^2}\cos(q_1) & \cos^2(q_1)q_2 \\ * & * & 2\cos(q_1) \end{bmatrix}.$$

The determinant of this matrix equals

$$2\lambda(1+q_2^2)c^3(q_1) \left[ 4\lambda - \frac{q_2^2}{\sqrt{1+q_2^2}} \right]$$

from which the claim follows immediately.  $\blacksquare$

## VI. ASYMPTOTIC STABILITY OF IDA-PBC DESIGNS WITH I&I OBSERVERS

In this section the stability properties of the combination of the IDA-PBC proposed in [7] (see also [6]), with the I&I observer derived in Section III, is studied. In particular, it is shown that the measurement of momenta,  $p$ , required in IDA-PBC, can be replaced by its observed signal,  $\hat{p}$ , preserving asymptotic stability of the desired equilibrium. In [6] a similar property is established for an IDA-PBC controller with a different I&I observer for the case of systems with under-actuation degree one written in *Spong's normal form* [24]—see Section 6 of [6]. It should be mentioned that to transform a mechanical system to Spong's normal form it is necessary, in general, to feed-back the full state, hence the result is not applicable for the problem at hand.

Even though global exponential convergence of the I&I observer has been established and, furthermore, mechanical systems are linear in  $u$ , the proof of this claim, in its global formulation, is non-trivial for the following reasons. First, the control law of IDA-PBC is quadratic in  $p$  and will, in general, depend on all the elements of this vector. Second, non-positivity of the Lyapunov function derivative is obtained in IDA-PBC via damping injection, more precisely, feeding-back the passive output, which is a function only of the actuated components of  $p$ , that is, the elements in the image of the input matrix  $G$ . Consequently, when  $p$  is replaced by their estimates the derivative of the (state-feedback) Lyapunov function will contain sign indefinite terms. While classical perturbation arguments allows to conclude local asymptotic stability, to establish the global version some particular properties of cascaded systems must be invoked.

For the sake of brevity the IDA-PBC methodology is not reviewed here, only the key equations needed for the analysis are given. The reader is referred to [6] and [7] for additional details. The objective in IDA-PBC is to assign to the closed-loop the energy function

$$H_d(q, p) = \frac{1}{2} p^\top M_d^{-1}(q) p + V_d(q) - V_d(q_\star)$$

where  $M_d = M_d^\top \in \mathbb{R}^{n \times n}$ ,  $V_d$  are the desired inertia matrix and potential energy function, respectively, and  $q_\star$  is the desired position, by preserving the mechanical structure of the system. This is achieved imposing the closed-loop dynamics

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{bmatrix} 0 & M^{-1}M_d \\ -M_d M^{-1} & J_2 - GK_v G^\top \end{bmatrix} \begin{pmatrix} \frac{\partial H_d}{\partial q} \\ \frac{\partial H_d}{\partial p} \end{pmatrix} \quad (53)$$

where  $K_v = K_v^\top \in \mathbb{R}_P^{n \times n}$  is a damping injection matrix and  $J_2(q, p)$  is a skew-symmetric matrix of the form

$$\begin{bmatrix} 0 & p^\top \alpha_1(q) & p^\top \alpha_2(q) & \dots & p^\top \alpha_{n-1}(q) \\ -p^\top \alpha_1(q) & 0 & p^\top \alpha_n(q) & \dots & p^\top \alpha_{2n-3}(q) \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ -p^\top \alpha_{n-1}(q) & -p^\top \alpha_{2n-3}(q) & \dots & \dots & 0 \end{bmatrix} \quad (54)$$

where  $\alpha_i : \mathbb{R}^n \rightarrow \mathbb{R}^n$ ,  $i = 1, \dots, (n/2)(n-1)$ , are free functions.

If  $q_* = \arg \min V_d(q)$  then  $(q_*, 0)$  is a stable equilibrium of the closed loop with Lyapunov function  $H_d$  clearly verifying

$$\dot{H}_d = -p^\top M_d^{-1} G K_v G^\top M_d^{-1} p \leq -c_1 |\bar{p}|^2$$

where, to simplify the notation in the sequel, we have defined the function

$$\bar{p}(q, p) := G^\top(q) M_d^{-1}(q) p$$

and use the convention of denoting with  $c_i$  an (often unspecified) positive constant—in this case  $c_1 := \lambda_{\min}\{K_v\}$ . Stability will be asymptotic if  $\bar{p}$  is a *detectable* output for the closed-loop system (53).

The full-state measurement IDA-PBC is given by

$$u(q, p) = (G^\top G)^{-1} G^\top \times \left( \frac{\partial H}{\partial q} - M_d M^{-1} \frac{\partial H_d}{\partial q} + J_2 M_d^{-1} p \right) - K_v \bar{p} \quad (55)$$

which, as shown in [6], may be written in the form

$$u(q, p) = u_0(q) + \begin{bmatrix} p^\top A_1(q) p \\ \vdots \\ p^\top A_m(q) p \end{bmatrix} - K_v \bar{p} \quad (56)$$

where the vector  $u_0 : \mathbb{R}^n \rightarrow \mathbb{R}^n$  and the matrices  $A_i : \mathbb{R}^n \rightarrow \mathbb{R}^{n \times n}$  are functions of  $q$ . As will be shown below, establishing boundedness of  $A_i$ ,  $i = 1, \dots, m$ , will be critical for our analysis. Towards this end, we center our attention on the quadratic terms in  $p$  of (55) stemming from  $\partial H / \partial q$  and  $\partial H_d / \partial q$ , and introduce the following.<sup>6</sup>

**Assumption 3:** The matrices  $\partial M / \partial q_i$ ,  $\partial M_d / \partial q_i$  and  $G$  are bounded.

**Proposition 9:** Consider the system (1). Define the *position feedback* controller as  $u = u(q, \hat{p})$  with  $\hat{p}$  an estimate of  $p$  generated by the I&I observer (17). Assume  $\bar{p}(q, p)$  is a *detectable* output for the closed-loop system (53) and that Assumptions 1 and 2 are satisfied. Then there exists a neighborhood of the point  $(q^*, 0, \beta(q^*))$  such that all trajectories of the closed-loop system starting in this neighborhood are bounded and satisfy

$$\lim_{t \rightarrow \infty} (q(t), p(t), \eta(t)) = (q^*, 0, \beta(q^*)).$$

Furthermore, if Assumption 3 holds and the full state-feedback controller (56) ensures global asymptotic stability then the neighborhood is the whole space  $\mathbb{R}^{3n}$ , thus boundedness and convergence are *global*.

<sup>6</sup>From (54) it is clear that the term  $J_2 M_d^{-1} p$  is also quadratic in  $p$ . It will be shown below that Assumption 3 allows to establish a suitable bound for this term as well.

*Proof:* To carry out the proof the overall system is written as a cascade interconnection of the observer error subsystem  $\dot{z} = -\mathcal{A}z$  and the full state-feedback dynamics (53). We first write  $u(q, \hat{p}) = u(q, p) + \chi(q, p, z)$  where we define

$$\chi(q, p, z) := \sum_{i=1}^m [z^\top \Psi^{-1} A_i \Psi^{-\top} z + z^\top \Psi^{-1} (A_i + A_i^\top) p] e_i - K_v G^\top M_d^{-1} \Psi^{-\top} z. \quad (57)$$

The overall system can then be written in the cascaded form

$$\begin{pmatrix} \dot{q} \\ \dot{p} \end{pmatrix} = \begin{bmatrix} 0 & M^{-1} M_d \\ -M_d M^{-1} & J_2 - G K_v G^\top \end{bmatrix} \begin{pmatrix} \frac{\partial H_d}{\partial q} \\ \frac{\partial H_d}{\partial p} \end{pmatrix} + \begin{bmatrix} 0 \\ G \end{bmatrix} \chi$$

$$\dot{z} = -\mathcal{A}z. \quad (58)$$

Note that the system with  $\chi = 0$  is asymptotically stable. Furthermore, the disturbance term is such that

$$G(q) \chi(q, p, 0) = 0.$$

Invoking well-known results of asymptotic stability of cascaded systems [25] completes the proof of local asymptotic stability. To establish the global claim we invoke the fundamental result of [26], see also [27], and see that the proof will be completed if we can establish boundedness of the trajectories  $(q(t), p(t))$ . Computing the time derivative of  $H_d$  along the trajectories of (58) we get the bound

$$\dot{H}_d \leq -c_1 |\bar{p}|^2 + |\bar{p}| |G \chi|. \quad (59)$$

From the expression above it is clear that the key step to prove boundedness of trajectories is to establish a suitable bound for  $|G \chi|$ . At this point Assumption 3 is imposed. Comparing (55) with (56) we observe that the matrices  $A_i$  will be bounded if Assumption 3 holds and  $J_2$  may be bounded as  $\|J_2\| \leq c_2 |p|$ . Now, from the IDA-PBC procedure we have that  $J_2$  satisfies the so-called kinetic energy PDE

$$G^\perp \left\{ \frac{\partial}{\partial q} (p^\top M^{-1} p) - M_d M^{-1} \frac{\partial}{\partial q} (p^\top M_d^{-1} p) + 2 J_2 M_d^{-1} p \right\} = 0.$$

Comparing in this equation the terms which are quadratic in  $p$  and (54) we conclude that, under Assumption 3,  $J_2$  will satisfy the bound above and the matrices  $A_i$  are also bounded.

From the previous discussion, and boundedness of  $z$ , we get the bound  $|G \chi| \leq |z| (c_2 + c_3 |p|)$ , which replaced in (59) yields

$$\dot{H}_d \leq -c_1 |\bar{p}|^2 + |\bar{p}| |z| (c_2 + c_3 |p|). \quad (60)$$

Now, invoking standard (Young's inequality) arguments we get

$$|\bar{p}| |z| \leq \frac{c_1}{c_2} |\bar{p}|^2 + \frac{c_2}{4c_1} |z|^2.$$

We replace this bound in the second right hand term in (60) to get

$$\begin{aligned} \dot{H}_d &\leq -c_1 |\bar{p}|^2 + \left[ \frac{c_1}{c_2} |\bar{p}|^2 + \frac{c_2}{4c_1} |z|^2 \right] c_2 + c_3 |\bar{p}| |z| |p| \\ &\leq \frac{c_2^2}{4c_1} |z|^2 + c_5 |z| |p|^2 \end{aligned}$$

where we have used the bound of  $|\bar{p}| \leq c_4|p|$  to define  $c_5 := c_3c_4$ . Now, let us consider the non-negative function

$$W(q, p, z) := H_d(q, p) + \frac{c_2^2 \bar{\lambda}(Q)}{4c_1\epsilon} \mathcal{V}(z)$$

where  $\mathcal{V}(z)$  is given in (19), which as shown in the proof of Proposition 1 verifies (20). Finally, evaluating the derivative of  $W$  we get

$$\dot{W} \leq c_5|z||p|^2 \leq \frac{2c_5}{\bar{\lambda}(M_d)}|z|W \quad (61)$$

where we have used the bounds  $W \geq H_d \geq (1/2)\bar{\lambda}(M_d)|p|^2$  to obtain the last inequality. Since  $z$  is clearly an *integrable* function, invoking the Comparison Lemma [28], we immediately conclude boundedness of  $W$  and, consequently, boundedness of the trajectories  $(q(t), p(t))$  and complete the proof. ■

## VII. SIMULATION RESULTS

The theoretical results of the previous sections have been verified through simulations of the inverted pendulum example. The dynamical equations for this system are given by (1), (2) with

$$M^{-1} = \frac{1}{m_3 - b^2 \cos^2 q_1} \begin{bmatrix} m_3 & -b \cos q_1 \\ * & 1 \end{bmatrix}, \quad V = a \cos q_1$$

$$G = e_2, \quad a = \frac{g}{l}, \quad b = \frac{1}{l}, \quad m_3 = \frac{M + m}{ml^2}$$

where  $q_1$  denotes the pendulum angle with respect to the upright vertical,  $q_2$  the cart position,  $m$  and  $l$  are, respectively, the mass and length of the pendulum,  $M$  is the mass of the cart and  $g$  is the gravitational acceleration. The equilibrium to be stabilized is the upward position of the pendulum ( $q_{1*} = 0$ ) with the cart placed in any desired location (arbitrary  $q_{2*}$ ).

The detailed expressions of the full-state IDA-PBC, given by (55), may be found in [13]. The proposed “certainty-equivalent” controller is obtained replacing  $p$  by  $\hat{p}$ , which is generated by the I&I observer

$$\begin{aligned} \dot{\eta}_1 &= \frac{\Lambda_{11}\sqrt{m_3}}{\sqrt{m_3 - b^2 \cos^2 q_1}}(\beta_1 - \eta_1) - \frac{a\sqrt{m_3} \sin q_1}{\sqrt{m_3 - b^2 \cos^2 q_1}} \\ &\quad + \frac{b \cos q_1}{\sqrt{m_3}\sqrt{m_3 - b^2 \cos^2 q_1}} u \\ \dot{\eta}_2 &= \frac{\Lambda_{22}}{\sqrt{m_3}}(\beta_2 - \eta_2) - \frac{1}{\sqrt{m_3}} u \\ \hat{p}_1 &= \frac{\sqrt{m_3 - b^2 \cos^2 q_1}}{\sqrt{m_3}}(\beta_1 - \eta_1) + \frac{b \cos q_1}{\sqrt{m_3}}(\beta_2 - \eta_2) \\ \hat{p}_2 &= \sqrt{m_3}(\beta_2 - \eta_2) \end{aligned}$$

with  $\beta$  given by (51). The OED takes the form

$$\begin{aligned} \dot{z}_1 &= -\frac{\Lambda_{11}\sqrt{m_3}}{\sqrt{m_3 - b^2 \cos^2 q_1}} z_1 \\ \dot{z}_2 &= -\frac{\Lambda_{22}}{\sqrt{m_3}} z_2 \end{aligned}$$

from which it is clear that the rate of convergence is (essentially) determined by the constant  $\Lambda_{11}$  and  $\Lambda_{22}$ .

TABLE I  
SIMULATION PARAMETERS FOR THE INVERTED PENDULUM EXAMPLE

$q_{2*} = 20$	$q_1(0) = \frac{\pi}{2} - 0.2$
$a = m_3 = 1$	$q_2(0) = -0.1$
$b = 1/g$	$p_1(0) = 0.1$
$\Lambda_{11} = \Lambda_{22} = 1$	$p_2(0) = 0.2$
$K_v = k = m_{22}^0 = 0.01$	$\eta_1(0) = \Lambda_{11} q_1(0)$
$P = 1$	$\eta_2(0) = \Lambda_{22}(q_2(0) + \frac{b \sin(q_1(0))}{m_3})$

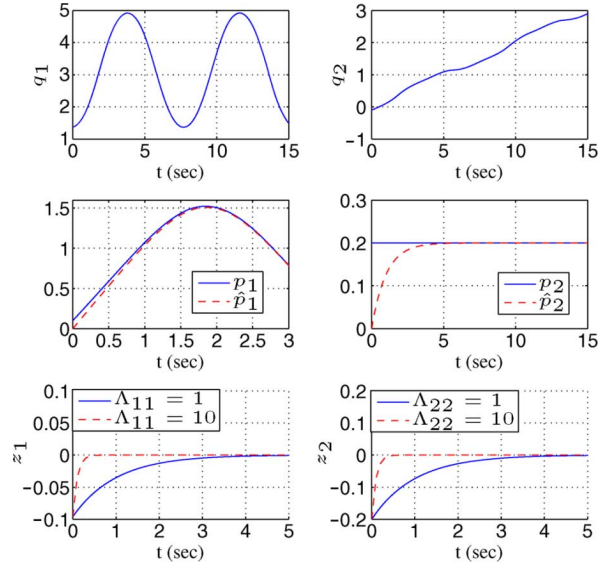


Fig. 6. Immersion and invariance observer for the open-loop system ( $u = 0$ ).

The values of the system and controller parameters, as well as the initial conditions, are shown in Table I. The initial conditions of the observer states  $(\eta_1(0), \eta_2(0))$  are chosen so that the initial estimate  $\hat{p}(0) = 0$ , that is, no prior knowledge for the initial momentum.

Simulation results are shown for the open-loop system, i.e.,  $u = 0$ , in Fig. 6. To reveal the role of the observer tuning gains, the time histories of  $z$  are depicted for  $\Lambda_{11} = \Lambda_{22}$  for the values 1 and 10. Fig. 7 shows the behavior of the system in closed loop with the IDA-PBC controller with full-state feedback and observer-based feedback. As it can be seen, the trajectories of the observer-based feedback system show an almost identical behavior with the trajectories of the full-state feedback system, concluding the effectiveness of the proposed scheme.

For the sake of comparison, the full-order I&I observer with dynamic scaling of Section III-C was also designed for this example. Details of its derivation are given in Appendix E. It should be underscored that, as indicated in Remark 4 and clearly illustrated in this example, this observer is more complex than the reduced order observer given above. Furthermore, as discussed in Appendix E, the need to obtain an integrable expression  $\tilde{\beta}$  makes the choice the matrix  $\mathcal{N}$  a non-trivial task. Simulations were carried out also for this observer yielding similar results with the proposed observer.

## VIII. CONCLUSION

A class of mechanical systems for which a globally exponentially stable reduced order observer can be designed has been identified in this paper. The class consists of all systems that can be rendered linear in (the unmeasurable) momenta via a (partial) change of coordinates  $P = \Psi(q)p$  and is characterized by (the solvability of) a set of PDEs. A detailed analysis

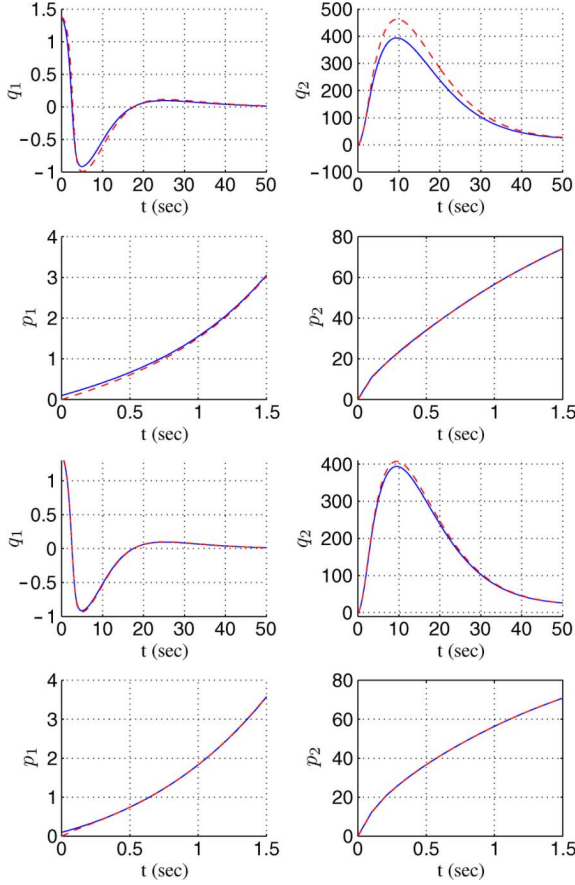


Fig. 7. Full-state (solid line) and observer-based (dashed line) IDA-PBC for  $\Lambda_{11} = \Lambda_{22} = 1$  and  $\Lambda_{11} = \Lambda_{22} = 10$ .

of the class is carried out and it is shown to contain many interesting practical examples and be much larger than the one reported in the literature of observer design and linearization. It is also proven that, under a very weak assumption, the observer can be used in conjunction with a globally asymptotically stabilizing full state-feedback IDA-PBC preserving *global* stability.

Several open questions are currently under investigation:

- The solvability of the PDEs arising in Assumption 1 is a widely open question. These PDEs are, in general, non-linear and quite involved. They are shown to be solvable for the robotic leg system and not-solvable for the classical ball-and-beam.
- It is possible to show that manipulators with more than one rotational joint are not Euclidean, that is, their mass matrix does not belong to  $\mathcal{S}_{ZRS}$ . However, it is not clear whether they belong to  $\mathcal{S}_T$ , or the larger set  $\mathcal{S}_{PLVCC}$ .
- In Remark 8 the difference between our observer design and the one used in [14] is discussed. Namely, the incorporation of the term  $\tilde{C}(q, \dot{q})$  in the observer, which is absent in our design. Some preliminary calculations show that, as expected, adding this term modifies the perturbing term  $\mathbf{D}_T$  leading to alternative conditions for it to be zero. The price to be paid is that, now, the stability analysis cannot be made systematic as done in Section IV.
- As we had seen, Euclidean systems are mechanical systems for which there exist coordinates in which the equations of motion become linear (see equation (45) and also Remark 5). Equivalently, a mechanical system is Euclidean if and only if the Riemann symbols of the

inertia matrix are identically zero. Similar to the characterization of Euclidean systems, it would be interesting to characterize the class of PLVCC systems, that is, derive necessary and sufficient conditions on the inertia matrix to verify the skew-symmetry conditions (3), (4).

## APPENDIX A TWO KEY LEMMATA

*Lemma 1:* Define the  $n \times n$  matrix

$$\mathcal{J} := \sum_{i=1}^n \left\{ \left( p^\top \frac{\partial \Psi}{\partial q_i} \right)^\top (e_i^\top \Psi) - (\Psi^\top e_i) \left( p^\top \frac{\partial \Psi}{\partial q_i} \right) \right\}. \quad (62)$$

Then

$$\mathcal{J}_{jk} = -p^\top [\Psi_j, \Psi_k]. \quad (63)$$

*Proof:* The proof is established simply computing the  $(jk)$ -th element of  $\mathcal{J}$  as

$$\begin{aligned} e_j^\top \mathcal{J} e_k &= \sum_{i=1}^n \left\{ \left( p^\top \frac{\partial \Psi}{\partial q_i} e_j \right) \Psi_{ik} - \left( p^\top \frac{\partial \Psi}{\partial q_i} e_k \right) \Psi_{ij} \right\} \\ &= p^\top \sum_{i=1}^n \left\{ \frac{\partial \Psi_j}{\partial q_i} \Psi_{ik} - \frac{\partial \Psi_k}{\partial q_i} \Psi_{ij} \right\} \\ &= p^\top \left\{ \frac{\partial \Psi_j}{\partial q} \Psi_k - \frac{\partial \Psi_k}{\partial q} \Psi_j \right\} \\ &= -p^\top [\Psi_j, \Psi_k]. \end{aligned}$$

*Lemma 2:* Define the  $n \times n$  matrices

$$\bar{J}_i := \sum_{j=1}^n [\Psi_i, \Psi_j] \Psi_j^\top (\Psi \Psi^\top)^{-1} M^{-1}, \quad i \in \bar{n}.$$

Then

$$\sum_{i=1}^n (\Psi^\top e_i) \left( p^\top \frac{\partial \Psi}{\partial q_i} \right) \Psi^{-1} M^{-1} p - \dot{\Psi}^\top p = \sum_{i=1}^n e_i (p^\top \bar{J}_i p).$$

*Proof:* We first note that

$$\begin{aligned} \dot{\Psi}^\top p &= \sum_{i=1}^n \left( \frac{\partial \Psi^\top}{\partial q_i} p \right) (e_i^\top M^{-1} p) \\ &= \sum_{i=1}^n \left( p^\top \frac{\partial \Psi}{\partial q_i} \right)^\top (e_i^\top \Psi) \Psi^{-1} M^{-1} p. \end{aligned}$$

Replacing (62) of Lemma 1 we obtain

$$\dot{\Psi}^\top p - \sum_{i=1}^n (\Psi^\top e_i) \left( p^\top \frac{\partial \Psi}{\partial q_i} \right) \Psi^{-1} M^{-1} p = \mathcal{J} \Psi^{-1} M^{-1} p.$$

Computing the  $i$ -th element of the vector completes the proof

$$\begin{aligned} e_i^\top \mathcal{J} \Psi^{-1} M^{-1} p &= e_i^\top \mathcal{J} \Psi^\top (\Psi \Psi^\top)^{-1} M^{-1} p \\ &= \sum_{k=1}^n \mathcal{J}_{ik} e_k^\top \Psi^\top (\Psi \Psi^\top)^{-1} M^{-1} p \\ &= - \sum_{k=1}^n p^\top [\Psi_i, \Psi_k] \Psi_k^\top (\Psi \Psi^\top)^{-1} M^{-1} p \\ &= -p^\top \bar{J}_i p \end{aligned}$$

where we have used (12) to obtain the third identity.  $\blacksquare$

# APPENDIX B PROOF OF (11)

We first note that  $\mathbf{D}_\Psi$  can be written as

$$\begin{aligned}\mathbf{D}_\Psi &= \sum_{i=1}^n \left[ (\Psi^\top e_i) \left( p^\top \frac{\partial \Psi}{\partial q_i} \right) - \left( \frac{\partial \Psi^\top}{\partial q_i} p \right) (e_i^\top \Psi) \right] \frac{\partial \bar{H}}{\partial P} \\ &\quad + \frac{1}{2} \sum_{i=1}^n (\Psi^\top e_i) \left[ P^\top \frac{\partial}{\partial q_i} \left( (\Psi^\top M \Psi)^{-1} \right) P \right] \\ &= -\mathcal{J} \frac{\partial \bar{H}}{\partial P} + \frac{1}{2} \sum_{i=1}^n (\Psi^\top e_i) \left[ P^\top \frac{\partial}{\partial q_i} \left( (\Psi^\top M \Psi)^{-1} \right) P \right]\end{aligned}$$

where we used the following identity:

$$\begin{aligned}\frac{\partial}{\partial q} \left\{ \frac{1}{2} p^\top M^{-1} p \right\} &= \sum_{i=1}^n e_i \left\{ p^\top \frac{\partial \Psi}{\partial q_i} \Psi^{-1} M^{-1} p \right. \\ &\quad \left. + \frac{1}{2} p^\top \Psi \frac{\partial}{\partial q_i} \left( (\Psi^\top M \Psi)^{-1} \right) \Psi^\top p \right\} \\ &= \sum_{i=1}^n e_i \left\{ p^\top \frac{\partial \Psi}{\partial q_i} (\Psi^\top M \Psi)^{-1} P \right. \\ &\quad \left. + \frac{1}{2} P^\top \frac{\partial}{\partial q_i} \left( (\Psi^\top M \Psi)^{-1} \right) P \right\} \\ &= \sum_{i=1}^n e_i \left\{ p^\top \frac{\partial \Psi}{\partial q_i} \frac{\partial \bar{H}}{\partial P} + \frac{1}{2} P^\top \frac{\partial}{\partial q_i} \left( (\Psi^\top M \Psi)^{-1} \right) P \right\}\end{aligned}$$

and the definition of  $\mathcal{J}$  given by (62) in Lemma 1. Replacing the expression of  $\mathbf{D}_\Psi$  in (6) we finally obtain

$$\begin{aligned}\dot{P} &= \mathcal{J} \frac{\partial \bar{H}}{\partial P} - \frac{1}{2} \sum_{i=1}^n (\Psi^\top e_i) \left[ P^\top \frac{\partial}{\partial q_i} \left( (\Psi^\top M \Psi)^{-1} \right) P \right] \\ &\quad - \Psi^\top \left( \frac{\partial V}{\partial q} - Gu \right) \\ &= \mathcal{J} \frac{\partial \bar{H}}{\partial P} - \Psi^\top \frac{\partial \bar{H}}{\partial q} + \Psi^\top Gu\end{aligned}$$

which corresponds to (11).

# APPENDIX C CONSTRUCTION OF THE EXAMPLE TO PROVE $M \in \mathcal{S}_T \not\Rightarrow M \in \mathcal{S}_{ZRS}$

First, we observe that (35) is a sufficient condition for skew-symmetry of  $\mathcal{B}_{(i)}$ . Condition (35) is satisfied by the vectors  $T_i = A_i x$  where  $x \in \mathbb{R}^3$  and  $A_i \in \mathbb{R}^{3 \times 3}$  are the rotation matrices

$$A_i = \begin{bmatrix} 0 & \Omega_{3i} & \Omega_{2i} \\ -\Omega_{3i} & 0 & \Omega_{1i} \\ -\Omega_{2i} & -\Omega_{1i} & 0 \end{bmatrix}, \quad i \in \{1, 2, 3\}$$

where  $\Omega_{jk} := e_j^\top e_k$ . However, the resulting matrix  $T = [T_1 | T_2 | T_3]$  has zero determinant, hence cannot qualify as a factor of  $M^{-1}$ .

<sup>7</sup>For reasons that will become clear below we find convenient to, temporarily, use the notation  $x$  instead of  $q$ .

To complete the example some concepts from Lie group theory see, e.g., [16], [29], must be invoked. The first observation is that the matrices  $A_i$  are tangent vectors at the identity point of the Lie group  $SO(3)$  and, furthermore, form a basis for its associated Lie algebra  $so(3)$ . We then extend these vectors to left-invariant vector fields on the group  $SO(3)$  using a push-forward of the left multiplication map  $L_g(h) = gh$ , where  $g, h \in SO(3)$ . The push-forward is defined as  $(L_g)_*(A_i) = gA_i$ , where  $g$  is taken to be the product matrix  $R(x) = R_1(x)R_2(x)R_3(x)$  with

$$\begin{aligned}R_1 &= \begin{bmatrix} \cos x_1 & \sin x_1 & 0 \\ -\sin x_1 & \cos x_1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, R_2 = \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos x_2 & \sin x_2 \\ 0 & -\sin x_2 & \cos x_2 \end{bmatrix} \\ R_3 &= \begin{bmatrix} \cos x_3 & \sin x_3 & 0 \\ -\sin x_3 & \cos x_3 & 0 \\ 0 & 0 & 1 \end{bmatrix}\end{aligned}$$

which is a parametrization (using the Euler angles) of  $SO(3)$ . The question is then to find the vectors  $\tilde{T}_i$ , whose push-forward by  $R_*$ , that is  $R_*(\tilde{T}_i)$ , will equal  $(L_R)_*(A_i)$ . This leads to the following set of equations:

$$\frac{\partial R}{\partial x_1} \tilde{T}_{i1}(x) + \frac{\partial R}{\partial x_2} \tilde{T}_{i2}(x) + \frac{\partial R}{\partial x_3} \tilde{T}_{i3}(x) = R(x) A_i, \quad i = 1, 2, 3.$$

Solving these equations one obtains the matrix  $\tilde{T}$

$$\tilde{T}(x) = \begin{bmatrix} -\sin(x_1) \cot(x_2) & -\cos(x_1) \cot(x_2) & 1 \\ \cos(x_1) & -\sin(x_1) & 0 \\ \sin(x_1) \operatorname{cosec}(x_2) & \cos(x_1) \operatorname{cosec}(x_2) & 0 \end{bmatrix}.$$

Some simple computations show that the matrix  $\tilde{T}$  has full rank (almost everywhere) and verifies (35) as desired.

The matrix  $\tilde{T}$  above has a singularity at zero that can be easily “removed” introducing a homeomorphism  $F : \mathbb{R} \times (0, \pi) \times \mathbb{R} \rightarrow \mathbb{R}^3 : x \mapsto q$ . For instance,  $F(x) = [x_1, \tan(x_2 - (\pi/2)), x_3]^\top$ , which has an inverse map  $F^I : \mathbb{R}^3 \rightarrow \mathbb{R}^3$ ,  $F^I(q) = [q_1, (\pi/2) + \tan^{-1}(q_2), q_3]^\top$ . Define the transformed vectors

$$T_i(q)f = \left[ \frac{\partial F}{\partial x}(x) \tilde{T}_i(x) \right]_{x=F^I(q)}, \quad i = 1, 2, 3$$

that, after some simple calculations, yields (34).

# APPENDIX D PROOF OF PROPOSITION 6

We begin by denoting  $\mathcal{C} \subset \mathbb{R}^n$  as the  $n$ -dimensional manifold defined by the configuration space of the generalized position coordinates  $q$ . Then, each  $T_i(q)$ ,  $i \in \bar{n}$  would be a vector field acting on the manifold  $\mathcal{C}$ . Since, the matrix  $T(q)$  has a full rank for all  $q$ , its columns are linearly independent. We now assume that:

- the columns of  $T(q)$  satisfy  $[T_i, T_j] = 0$ ,  $i, j \in \bar{n}$  and  $q \in \mathcal{C}$ .
- the  $n$  vector fields  $T_i$ ,  $i \in \bar{n}$  are complete, that is, the integral curves of the vector fields exist for all times  $t$ .

Then, from Theorem 2.36 in [30], we know that there exists a coordinate chart for  $\mathcal{C}$  given by the coordinates  $\bar{q} = Q(q)$  for some  $Q : \mathbb{R}^n \rightarrow \mathbb{R}^n$  such that, the vector fields in the new coordinates satisfy  $T_i(\bar{q}) = e_i$  where  $e_i$  denotes the  $i^{\text{th}}$  natural basis

vector of  $\mathbb{R}^n$ . Further, this coordinate chart would be global and hence the mapping from  $q$  to  $Q$  is bijective for all  $q \in \mathcal{C}$ .

We next invoke the fact that the vector fields transform in a covariant fashion [30] under such coordinate changes which means

$$T_i(\bar{q}) = \frac{\partial Q}{\partial q}(q) T_i(q) = e_i. \quad (64)$$

Subsequently, we perform the following computations

$$\frac{\partial Q}{\partial q}(q) T_i(q) = e_i \quad (65)$$

$$\Rightarrow \frac{\partial Q}{\partial q}(q) [T_1(q) | T_2(q) | \dots | T_n(q)] = I_{n \times n} \quad (66)$$

$$\Rightarrow \frac{\partial Q}{\partial q} = T^{-1}(q). \quad (67)$$

Thus, we can conclude that if the columns of  $T(q)$  commute, then  $T^{-1}(q)$  is the Jacobian of some vector  $S(q)$  and is thus integrable. We now assume that  $T^{-1}(q) = \partial S / \partial q$  for some vector  $Q : \mathbb{R}^n \rightarrow \mathbb{R}^n$ . We then easily obtain,  $(\partial Q / \partial q)(q) T_i(q) = e_i$  which implies that there exists a set of coordinates  $\bar{q} = Q(q)$  such that in those coordinates, the columns of  $T$  assume the form  $T_i(\bar{q}) = \partial / \partial \bar{q}_i$ . We once again invoke Theorem 2.36 in [30] and conclude that the columns of  $T(q)$  commute among each other. Hence, the proof follows.

#### APPENDIX E

##### CALCULATIONS FOR THE FULL ORDER OBSERVER OF THE INVERTED PENDULUM EXAMPLE

For the inverted pendulum on the cart example, we have to choose  $\mathcal{N}$  such that

$$\mathcal{N}T + (\mathcal{N}T)^\top \geq \epsilon I. \quad (68)$$

Subsequently, one computes

$$\tilde{\beta}(q, \hat{q}) = \int \mathcal{N}_1(q_1, \hat{q}_2) dq_1 + \int \mathcal{N}_2(\hat{q}_1, q_2) dq_2. \quad (69)$$

It is interesting to note that the natural choices  $\mathcal{N} = T^{-1}$  or  $\mathcal{N} = T^\top$ , with  $T$  the lower triangular Cholesky factor (31) lead to elliptic integral expressions for  $\tilde{\beta}$ . Thus, keeping  $T$  as the Cholesky factor in (31), we choose

$$\mathcal{N} = \rho \begin{bmatrix} m_3 - b^2 \cos^2 q_1 & 0 \\ \frac{(1+q_2^2)b \cos q_1}{m_3} & 1 + q_2^2 \end{bmatrix} \quad (70)$$

where  $\mathcal{N}$  is non-integrable, yielding

$$\mathcal{N}T = \rho \begin{bmatrix} \sqrt{m_3} \sqrt{m_3 - b^2 \cos^2 q_1} & 0 \\ 0 & \frac{1+q_2^2}{\sqrt{m_3}} \end{bmatrix}.$$

Hence (68) gets satisfied with  $\epsilon > 0$  and with  $\rho = \epsilon/k_1$ ,  $k_1 = 2 \min\{1/\sqrt{m_3}, \sqrt{m_3}/\sqrt{m_3 - b^2}\}$ . Next, compute  $\tilde{\beta}(q, \hat{q})$  using (69) and (70) as

$$\tilde{\beta}(q, \hat{q}) = \rho \begin{bmatrix} \left(m_3 - \frac{b^2}{2}\right) q_1 - \frac{b^2}{4} \sin(2q_1) \\ (1 + \hat{q}_2^2) \frac{b \sin q_1}{m_3} + q_2 + \frac{q_2^3}{3} \end{bmatrix} \quad (71)$$

which yields

$$\begin{aligned} \frac{\partial \tilde{\beta}}{\partial q} &= \rho \begin{bmatrix} m_3 - b^2 \cos^2 q_1 & 0 \\ \frac{(1+\hat{q}_2^2)b \cos q_1}{m_3} & 1 + q_2^2 \end{bmatrix} \\ \frac{\partial \tilde{\beta}}{\partial \hat{q}} &= \rho \begin{bmatrix} 0 & 0 \\ 0 & \frac{2\hat{q}_2 b \sin q_1}{m_3} \end{bmatrix}. \end{aligned}$$

We then have

$$\mathcal{N}_1(q_1, \hat{q}_2) = \mathcal{N}_1(q_1, q_2) - \delta_{11} \tilde{q}_1 - \delta_{12} \tilde{q}_2$$

$$\mathcal{N}_2(\hat{q}_1, q_2) = \mathcal{N}_2(q_1, q_2) - \delta_{21} \tilde{q}_1 - \delta_{22} \tilde{q}_2$$

from which one obtains that

$$\delta_{11} = \delta_{21} = \delta_{22} = 0, \delta_{12} = \rho \begin{bmatrix} 0 \\ \frac{-b \cos q_1}{m_3} (2q_2 + \tilde{q}_2) \end{bmatrix}.$$

Subsequently

$$\Delta_1 = 0, \Delta_2 = \rho \begin{bmatrix} 0 & 0 \\ \frac{-b \cos q_1}{m_3} (2q_2 + \tilde{q}_2) & 0 \end{bmatrix}.$$

Hence

$$\|\Delta_1\| = 0, \quad \|\Delta_2\| = \left| \frac{\rho b \cos q_1}{m_3} (2q_2 + \tilde{q}_2) \right|$$

and further

$$K(q, \tilde{q}, r) = cr^2 \begin{bmatrix} 1 & 0 \\ 0 & 1 + \rho^2 \frac{b^2 \cos^2 q_1}{m_3^2} (2q_2 + \tilde{q}_2)^2 \end{bmatrix}.$$

Finally

$$\dot{r} = -\epsilon(r-1) + \frac{cr\rho^2 \tilde{q}_2^2 b^2 \cos^2 q_1}{m_3^2} (2q_2 + \tilde{q}_2)^2, r(0) = 1.$$

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**Aneesh Venkatraman** was born in Tiruchirappalli, India, in 1982. He received the B.Tech. and M.Tech. degrees in aerospace engineering with specialization in dynamics and controls from the Indian Institute of Technology, Bombay, India in 2005, and is currently pursuing the Ph.D. degree at the Institute of Mathematics and Computing Science, University of Groningen, Groningen, The Netherlands.

He was a Guest Researcher at the Laboratoire des Signaux et Systèmes, Supelec, France (Sep–Oct 2007, Feb–Mar 2008) and at the Department of Elec-

trical and Electronic Engineering, Imperial College of London (Jan 22nd–Feb 7th 2009). His research interests are mainly in the fields of nonlinear control with focus on mechanical and electromechanical systems, robotics, optimal control with emphasis on path planning, and differential geometric control theory.



**Romeo Ortega** (F'99) was born in Mexico. He received the B.Sc. degree in electrical and mechanical engineering from the National University of Mexico, Mexico City, in 1974, the M.Eng. degree from the Polytechnical Institute of Leningrad, Leningrad, Russia, in 1978, and the Docteur D'Etat degree from the Politechnical Institute of Grenoble, Grenoble, France, in 1984.

He then joined the National University of Mexico, where he worked until 1989. He was a Visiting Professor at the University of Illinois, Urbana-Champaign, from 1987 to 1988, and at McGill University, Montreal, QC, Canada, from 1991 from 1992. He is currently in the Laboratoire de Signaux et Systèmes (SUPELEC), Paris, France. He is an Associate Editor of *Systems and Control Letters* and the *International Journal of Adaptive Control and Signal Processing*. His research interests are in the fields of nonlinear and adaptive control, with special emphasis on applications.

Dr Ortega was a Fellow of the Japan Society for Promotion of Science, from 1990 to 1991. He is an Editor at Large of the IEEE TRANSACTIONS ON AUTOMATIC CONTROL. He has been a member of the French National Researcher Council (CNRS) since June 1992. He has served as Chairman in several IFAC and IEEE committees and editorial boards. Currently he chairs the Automatica Paper Prize Award Committee (2009–2012).



**Ioannis Sarras** (M'09) was born in Athens, Greece, in 1982. He received the B.S. degree from the Automation Engineering Department, Technological Education Institute (T.E.I.), Piraeus, Greece, in 2004, the Master of Research (M2R) degree in automatic control from the Université Paul Sabatier, Toulouse III, France, in 2006, and is currently pursuing the PhD degree at the Laboratoire des Signaux et Systèmes, Université Paris-Sud 11, France.

His research interests are in the fields of nonlinear control with emphasis on mechanical systems.



**Arjan van der Schaft** (F'02) received the M.S. and Ph.D. degrees in mathematics from the University of Groningen, The Netherlands.

In 1982, he joined the University of Twente, The Netherlands, where he was appointed as a Full Professor in 2000. In 2005, he returned to Groningen as a Full Professor in Mathematics. He has served as Associate Editor for *Systems & Control Letters*, *Journal of Nonlinear Science*, *SIAM Journal on Control*. Currently he is Associate Editor for *Systems and Control Letters*, *Journal of Geometric Mechanics*, and Editor-at-Large for the *European Journal of Control*. He is co-author of a number of books, including *Nonlinear Dynamical Control Systems* (1990),  *$L_2$ -Gain and Passivity Techniques in Nonlinear Control* (1996, 2000), and *An Introduction to Hybrid Dynamical Systems* (2000). He was an Invited Speaker at the 2006 International Congress of Mathematicians, Madrid, Spain.

Dr. van der Schaft served as an Associate Editor with the IEEE TRANSACTIONS ON AUTOMATIC CONTROL.